FINITENESS CONDITIONS OF WREATH PRODUCTS OF SEMIGROUPS AND RELATED PROPERTIES OF DIAGONAL ACTS

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Declaration

I, Michael Robert Thomson, declare that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

Name: M. Robert Thomson **Signature:**.............................. **Date:** ....

I was admitted to the Faculty of Science of the University of St Andrews as a candidate for the degree of Doctor of Philosophy in September 1997.

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I certify that Michael Robert Thomson has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

Name: Edmund F. Robertson **Signature:**.............................. **Date:** ....
Name: Nik Ruškuc **Signature:**.............................. **Date:** ....

I agree that access to my thesis in the University of St Andrews shall be unrestricted.

Name: M. Robert Thomson **Signature:**.............................. **Date:** ....

Preface

Wreath products are one of the most important constructions both for groups and semigroups. Their significance stems from the Universal Embedding Theorem for groups, and the Krohn–Rhodes Decomposition Theorem for semigroups. The former asserts that for two groups, N and Q, every group extension of N by Q is isomorphic to a subgroup of $N \mathrm{wr} Q$; for details see for example [6]. The latter states that every finite semigroup is a homomorphic image of a subsemigroup of a wreath product of groups and group-free semigroups; for details see [8] or [16]. Wreath products have also been used for infinite groups, primarily as a tool for constructing examples and in embedding results; for example see [9, 10, 11].

Wreath products of infinite semigroups are encountered far less often in the literature. One reason for this is that various basic properties, such as generating sets, defining relations, and even membership (in the case of restricted wreath products), are difficult to determine. It is the main aim of this thesis to address some of these problems in a systematic way.

We start by introducing some basic definitions and results in Chapter 1, and introduce the wreath product itself in Section 4. In Chapter 2 we give necessary and sufficient conditions for the wreath product of two monoids to be finitely generated or finitely presented. It turns out that, slightly surprisingly, wreath products of monoids behave in exactly the same way as wreath products of groups with regard to finite presentability, but not with regard to finite generation.

Then we move our attention from monoids to semigroups, and in Chapter 3 consider the same questions of finite generation and finite presentability of $S\mathrm{wr}T$, in the special case where the top semigroup T is finite. The obtained results show unexpected connections with a certain class of semigroup actions, which themselves are investigated further in Chapter 6.

In Chapter 4 we look at wreath products in the case where the top semigroup is infinite. We show that S must be a monoid if $S\mathrm{wr}T$ is to be finitely generated in this case. We also obtain some technical necessary and sufficient conditions for $S\mathrm{wr}T$ to be finitely generated. We do not have a complete picture in this case, and there are opportunities for further research. Chapter 5 contains a selection of further results, concerning other finiteness conditions of wreath products. We also give some examples shedding further light on the results from the previous chapters. Finally, Chapter 6 develops a systematic theory of the so called diagonal acts, which have been encountered in Chapters 3 and 4, and makes some interesting connections with power semigroups.

The most important results from this thesis will be published in the research articles, [19, 17, 18]. However, this thesis is not simply a compilation of these papers. The results here have been presented in more detail, in many cases have been extended, and a number of supplementary results and examples are also included.

My interest in wreath products started after a lecture given by J.D.P. Meldrum in St Andrews. The work in this thesis owes much to his book, [16]. I would also like to thank my supervisors, Professor E.F. Robertson and Dr N. Ruškuc, for their ideas, guidance and support during my time in St Andrews. I am grateful to the E.P.S.R.C. for the financial support over the 3 years of my study.

This thesis is dedicated to my parents, whose support and encouragement throughout seven years of study have made this work possible.

Abstract

The purpose of this thesis is to consider finite generation, finite presentability and related properties of restricted wreath products of semigroups.

We show that the wreath product $A\text{wr}B$ of two monoids is finitely generated if and only if A and B are finitely generated and the action by right multiplication on B of the group of units of B has only finitely many orbits. Also we show that the wreath product $A\text{wr}B$ of two non-trivial monoids is finitely presented if and only if A is finitely presented and B is finite. The situation is more complicated in the case of the wreath product $S_e\text{wr}T$ of two semigroups with respect to an idempotent $e \in S$. We give a complete characterization for finite generation in the case where T is finite. This result depends on the properties of the diagonal action of S on $S \times S$. We also prove that if this action is not finitely generated, then $S_e\text{wr}T$ (with S infinite and T finite) is finitely presented if and only if $S \times S$ is finitely presented and T is the direct product of a monoid and a left zero semigroup. In the case where T is infinite, we prove that S must be a monoid in order for $S\text{wr}T$ to be finitely generated. We show that the finiteness properties of periodicity and local finiteness are preserved under the wreath product construction. We conclude the thesis with a systematic investigation into the properties of diagonal acts of semigroups, and make some interesting connections between diagonal acts and power semigroups.

Chapter 1

Definitions and Preliminary Results

In the following sections we will introduce the basic definitions and results that we will require throughout the rest of the thesis. Section 1 gives a brief introduction to Green's relations for semigroups. Section 2 introduces presentations and gives a couple of well known results that will prove useful later on. In Section 3 we look at three semigroup constructions, the direct product, the semi-direct product and the free product. Finally in Section 4, we introduce the wreath product construction, and give some preliminary results.

The results and examples throughout this thesis are numbered within each chapter. For example, Theorem 3.3 refers to the third result in Chapter 3. We use a black square, (■) to denote the end of a mathematical argument. A black square immediately following a result means that no proof will be given.

1 Green's Relations

Let S be a semigroup, and let $a \in S$. The smallest right ideal of S containing a is $aS \cup \{a\}$, usually denoted aS^1, where S^1 is the monoid obtained from S by adding an identity only if necessary. We call aS^1 the principal right ideal generated by a. We can define an equivalence relation $\mathcal{R}$ on S by the rule that $a\mathcal{R}b$ if and only if a and b generate the same principal right ideal, that is, $aS^1 = bS^1$. We will refer to $\mathcal{R}$-equivalence classes of S as $\mathcal{R}$-classes, and denote by R_a the $\mathcal{R}$-class of S containing a.

The inclusion order among principal right ideals of S induces a partial order among the $\mathcal{R}$-classes:

$$R_a \leqslant R_b \text{ if and only if } aS^1 \subseteq bS^1.$$

We will often write $a \leqslant_{\mathcal{R}} b$ when $aS^1 \subseteq bS^1$, and $a <_{\mathcal{R}} b$ when the inclusion $aS^1 \subset bS^1$ is strict.

By considering principal left ideals S^1a we may define the equivalence relation $\mathcal{L}$, and partial orders $\leqslant_{\mathcal{L}}$ and $<_{\mathcal{L}}$ on elements of S in the same way. We will write L_a to mean the $\mathcal{L}$-class of S containing a. More information about these Green's relations, and the $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ relations not defined here can be found in any standard semigroup theory textbook - see for example [12]. We conclude this section with the following well known theorem, which will be required in later chapters.

Theorem 1.1 *Let S be a semigroup containing a right identity e. If $s \geqslant_{\mathcal{R}} e$ for all $s \in S$ then S is a group.*

PROOF. For a proof, see [14, Proposition 1.3]. ∎

2 Presentations

Let A be an alphabet (a non-empty set). We denote by A^+ the semigroup of all finite strings over A under concatenation. This is the free semigroup on A. We define $A^* = A^+ \cup \{\varepsilon\}$ where ε denotes the 'empty string'. Then A^* is a monoid under concatenation of strings with identity element ε, and is known as the free monoid on A.

Definition 1.2 Let S be a semigroup. A congruence ρ on S is an equivalence relation on S with the additional property that for any $(x, y), (a, b) \in \rho$ we have $(xa, yb) \in \rho$.

It is easy to check that if ρ is a congruence on S then the equivalence classes S/ρ form a semigroup - a homomorphic image of S.

A semigroup presentation is a pair, written $\langle A \mid R \rangle$, where A is an alphabet and R is a subset of $A^+ \times A^+$. This presentation defines the semigroup A^+/ρ, where ρ is the smallest congruence on A^+ containing R. A monoid presentation is a pair $\langle B \mid Q \rangle$ where B is an alphabet and Q a subset of $B^* \times B^*$. This presentation defines the monoid B^*/σ, where σ is the smallest congruence on B^* containing Q. A semigroup or monoid S is finitely presented if there exists a finite semigroup or monoid presentation defining it. If S is defined by the presentation $\langle A \mid R \rangle$ then we may sometimes abuse notation and write $S = \langle A \mid R \rangle$. Note that a monoid M may be defined by either a semigroup presentation or a monoid presentation, and if M is defined by a finite monoid presentation then we can find a finite semigroup presentation defining M.

If S is defined by the presentation $\langle A \mid R \rangle$ then we may consider words over A as elements of S. We will adopt the following convention: if two words w_1 and w_2 over A are equal in A^+ we will write $w_1 \equiv w_2$. If they represent the same element of S (but are not necessarily equal in A^+) we will write $w_1 = w_2$. In particular, a pair (u,v) from R will usually be written as $u = v$. The following well known theorem tells us when two words over A are equal in S.

Theorem 1.3 *Suppose that the semigroup S is defined by a presentation $\langle A \mid R \rangle$. Two words w_1 and w_2 over A are equal in S if and only if there exists a sequence of words over A,*

$$w_1 \equiv \gamma_1, \gamma_2, \dots, \gamma_r \equiv w_2,$$

such that for each $1 \leqslant i < r$, the word γ_{i+1} may be obtained from γ_i by a single application of a relation from R. That is, there exist $\alpha_i, \beta_i \in A^*$ such that

$$\alpha_i u \beta_i \equiv \gamma_i$$
$$\alpha_i v \beta_i \equiv \gamma_{i+1}$$

for some relation (u,v) where either $(u,v) \in R$ or $(v,u) \in R$. Such a sequence is called an elementary sequence. ∎

For a proof see [12, Proposition 1.5.9]. The following theorems will be needed later in the thesis:

Theorem 1.4 *Let S be a finitely generated semigroup, and suppose that $S = \langle U \rangle$ where U is infinite. Then we may find a finite subset U' of U such that $S = \langle U' \rangle$.*

Proof. This is a standard theorem, but we give a short proof for completeness. Suppose that $S \cong \langle U \rangle \cong \langle X \rangle$ where U is infinite and $X = \{x_1 \dots x_r\}$ is finite. Let $\phi : \langle U \rangle \to \langle X \rangle$ be an isomorphism. For each x_i we find a word $u_{i1} \dots u_{ir_i} \in U^*$ such that $(u_{i1} \dots u_{ir_i})\phi = x_i$. Then the image under ϕ of $\langle U' \rangle$ where

$$U' = \{u_{ij} : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant r_i\}$$

is the whole of $\langle X \rangle$, and so $S \cong \langle U' \rangle$ as required. ∎

The corresponding theorem for finite presentability also holds:

Theorem 1.5 *Let S be a finitely presented semigroup, and suppose that S is defined by the presentation $\langle A \,|\, R \rangle$ where A is finite but R may be infinite. Then there exists a finite subset $R' \subseteq R$ such that S is defined by the presentation $\langle A \,|\, R' \rangle$.*

PROOF. This is an immediate consequence of [22, Proposition 1.3.1]. ∎

Theorem 1.6 *Let S be a finitely generated (finitely presented) semigroup and let T be a subsemigroup of S whose complement, $S \setminus T$, is an ideal of S. Then T is also finitely generated (finitely presented).*

PROOF. Let S be a semigroup, and let T be a subsemigroup whose complement $S \setminus T$ is an ideal of S. If S is generated by a set X then it is easy to see that T is generated by $X \wedge T$. Indeed, given $t \in T$ we may write $t = x_1 \ldots x_r$ as a product of generators from X. Each x_i must in fact lie in T since $S \setminus T$ is an ideal of S. Thus if S is finitely generated, T is also finitely generated.

Suppose that S is defined by the presentation $\langle X \,|\, R \rangle$, and that T is a subsemigroup of S whose complement $S \setminus T$ is an ideal of S. Let $X' = X \wedge T$ and

$$R' = \{(u, v) \in R : u \in T\}.$$

We claim that T is defined by the presentation $\langle X' \,|\, R' \rangle$. Indeed, given $t \in T$ we write $t = x_1 \ldots x_r$ as a product of generators from X. Each x_i must lie in T since $S \setminus T$ is an ideal of S, and so X' generates T. Given a relation $(u, v) \in R'$ we have $u \in T$, and so, again since $S \setminus T$ is an ideal, u and v must be words over X'. Each relation $u = v$ obviously holds in T. Conversely, given two words w_1 and w_2 in $(X')^*$ such that $w_1 = w_2$ in T, we must have $w_1 = w_2$ in S, and so we may use Theorem 1.3 to find an elementary sequence of words over X,

$$w_1 \equiv \gamma_1, \gamma_2, \ldots, \gamma_n \equiv w_2$$

such that for each $1 \leqslant i < n$, the word γ_{i+1} may be obtained from γ_i by a single application of a relation from R. Each γ_i lies in T, and so, since $S \setminus T$ is an ideal, the relation used to obtain the word γ_{i+1} from γ_i must lie in R'. So by Theorem 1.3, the presentation $\langle X' \mid R' \rangle$ defines T. In particular, if S is finitely generated (finitely presented) then T is finitely generated (finitely presented). ■

Theorem 1.7 (N. Ruškuc) *Let S be a semigroup and $T \leqslant S$ a subsemigroup, such that T has finite index in S, i.e. $|S \setminus T|$ is finite. Then S is finitely generated (finitely presented) if and only if T is finitely generated (finitely presented).*

For a proof of Theorem 1.7 see [23, Theorems 1.1 and 1.3].

3 Semigroup Constructions

Although the main emphasis of this thesis is the wreath product construction, we will have reason to use various properties of other semigroup constructions. In this section we introduce the *direct product*, the *semi-direct product* and the *free product* and give some preliminary results.

3.1 The Direct Product

Definition 1.8 Let S and T be semigroups. The *direct product* of S and T, denoted $S \times T$, is defined to be the cartesian product of S and T with componentwise multiplication:

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2) \quad (s_1, s_2 \in S,\ t_1, t_2 \in T).$$

It is clear that for any two semigroups S and T, $S \times T \cong T \times S$. If S and T are monoids defined by the (monoid) presentations $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$ respectively, then it is easily seen that the (monoid) presentation

$$\langle A, B \mid R, Q, ab = ba \ (a \in A, b \in B) \rangle$$

defines $S \times T$. It is also simple to show that the direct product $S \times T$ of two monoids is finitely presented (respectively finitely generated) if and only if both S and T are finitely presented (respectively finitely generated). The question of finite generation/presentability is much harder for two semigroups. For example, it is easy to see that the direct product of two free semigroups is never finitely generated. The following definitions and theorems are taken directly from [20]. Here we are using the notation S^2 to mean the set $\{s_1 s_2 : s_1, s_2 \in S\}$. This is not to be confused with $S \times S$, the direct product of S with itself.

Theorem 1.9 (Robertson, Ruškuc, Wiegold) *Let S and T be two infinite semigroups. Then $S \times T$ is finitely generated if and only if both S and T are finitely generated and $S^2 = S$ and $T^2 = T$.* ∎

Definition 1.10 Let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation, let S be the semigroup defined by it, and let $w_1, w_2 \in A^+$ be arbitrary words. The pair (w_1, w_2) is called a *critical pair* (for S with respect to $\mathcal{P}$) if the following conditions are satisfied:

(i) the relation $w_1 = w_2$ holds in S;

(ii) for every elementary sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_k \equiv w_2$ with respect to $\mathcal{P}$ there exists i $(1 \leqslant i \leqslant k)$ such that $|\alpha_i| < \min(|w_1|, |w_2|)$.

Definition 1.11 Let S be a semigroup with a finite generating set A. We say that S is *stable* (with respect to A) if there exists a finite presentation $\mathcal{P} = \langle A \mid R \rangle$, defining S in terms of A, with respect to which S has no critical pairs.

It is shown in [20] that stability is invariant under change of (finite) generating set.

Theorem 1.12 (Robertson, Ruškuc, Wiegold) *Let S and T be two infinite semigroups. The direct product $S \times T$ is finitely presented if and only if*

(i) $S^2 = S$ *and* $T^2 = T$;

(ii) S *and* T *are finitely presented and stable.* ∎

For more information on direct products of semigroups and stability, see either [2] or[20].

We define the direct product of a finite number of semigroups inductively. We simply define

$$S_1 \times S_2 \times \ldots \times S_{n-1} \times S_n = (S_1 \times S_2 \times \ldots \times S_{n-1}) \times S_n.$$

A special case of this, which we will make much use of in this thesis, is when the S_i are all equal to S. We will use the notation $S^{(n)}$ to mean the direct product of n copies of S. The following lemma gives us an alternate description of the direct product of copies of S.

Lemma 1.13 *Let $\mathcal{F}(n, S)$ be the set of all functions from the set $\{1, 2, \ldots, n\}$ into S. Given $f, g \in \mathcal{F}(n, S)$ we define a new function $fg \in \mathcal{F}(n, S)$ such that*

$$i(fg) = (if)(ig) \ (1 \leqslant i \leqslant n).$$

Then the semigroup $\mathcal{F}(n, S)$ with this componentwise multiplication is isomorphic to $S^{(n)}$.

Proof. It is clear that an element $s = (s_1, s_2, \ldots, s_n)$ of $S^{(n)}$, an n-tuple with components in S, uniquely defines a function $f : \{1, 2, \ldots, n\} \to S$. We simply define $if = s_i$ for all i. Conversely a function $f : \{1, 2, \ldots, n\} \to S$ uniquely defines the n-tuple $(1f, 2f, \ldots, nf)$. To show that this bijection is actually an isomorphism, simply observe that

$$(s_1, s_2, \ldots, s_n)(t_1, t_2, \ldots, t_n) = (s_1t_1, s_2t_2, \ldots, s_nt_n),$$

while if $if = s_i$, $ig = t_i$ then $i(fg) = s_it_i$. ∎

It is clear that if X is a set of size n then $\mathcal{F}(n, S)$ is isomorphic to S^X, the semigroup of all functions from X into S with componentwise multiplication. As a consequence of Lemma 1.13 we will alternate between viewing $S^X = S^{(n)}$ as a semigroup of n-tuples or a semigroup of functions as it suits us. In fact it is convenient to use the 'function' model to define the direct product of countably many copies of S.

Definition 1.14 Let S be a semigroup. The *unrestricted direct product* of countably many copies of S, denoted $S^{\mathbb{N}}$, is the semigroup of functions from $\mathbb{N}$ into S under the componentwise multiplication defined above.

It is clear that if X is any infinite countable set, then the semigroup S^X of all functions from X into S with componentwise multiplication is isomorphic to $S^{\mathbb{N}}$. Since $S^{\mathbb{N}}$ has size $|S|^{\mathbb{N}}$, if S is non-trivial, $S^{\mathbb{N}}$ is uncountable. Providing S has an idempotent element (usually taken to be the identity of S, if it exists), we may define a smaller direct product - a subsemigroup of the unrestricted direct product - using the notion of support, defined next:

Definition 1.15 Let S be a semigroup with fixed idempotent element e, let X be any set and let f be a function from X into S. The *support* of f (with respect to e), denoted $\operatorname{supp}(f)$, is defined to be the set $\{x \in X : xf \neq e\}$.

It is clear that since e is idempotent, $\operatorname{supp}(fg) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$. Indeed, if $x \notin \operatorname{supp}(f)$ and $x \notin \operatorname{supp}(g)$ then $x(fg) = (xf)(xg) = e^2 = e$. Therefore the (componentwise) product of two functions with finite support is again a function with finite support. Thus we may make the following definition:

Definition 1.16 Let S be a semigroup with a fixed idempotent element e. The *(restricted) direct product* of countably many copies of S, denoted $S_e^{(\mathbb{N})}$, is the semigroup of all functions from $\mathbb{N}$ into S with finite support (with respect to e), under the same componentwise multiplication as before.

Again it is clear that if X is a countably infinite set, then $S_e^{(\mathbb{N})}$ is isomorphic to $S_e^{(X)}$, the semigroup of all functions from X into S with finite support under the same componentwise multiplication. The set X is essentially an indexing set, and any structure other than its size is irrelevant.

We now introduce some notation that is used extensively throughout this thesis. Suppose for the moment that the fixed idempotent element is 1_S, the identity of S. We will write $\overline{s_x}$ to denote the element of $S^{(X)}$ which, considered as a function, maps x to s and $X \setminus \{x\}$ to 1_S. In other words,

$$y\overline{s_x} = \begin{cases} s & \text{if } y = x, \\ 1_S & \text{otherwise.} \end{cases}$$

It is clear that any function with finite support may be written as a finite product of such elementary functions with support of size one. Thus if S is generated by the set A, then $S^{(X)}$ is generated by the set $\{\overline{s_x} : x \in X, a \in A\}$. We will write $\overline{1}$ to denote the identity element of $S^{(X)}$, the function with empty support mapping the whole of X to 1_S.

We extend this notation to the more general case S_e (where $e \neq 1_S$) in the obvious way. In this case $\overline{e}$ will denote the function mapping the whole of X to e, and $\overline{s_x}$ will denote the function mapping x to s and everything else to e.

3.2 The Semi-direct Product

Definition 1.17 Let S and T be semigroups, and let $\phi : T \to \text{End}(S)$ be a homomorphism from T to the semigroup of endomorphisms acting on S on the left. For elements $s \in S$ and $t \in T$ we will denote by ${}^{t}s$ the element $(t\phi)s$ of S. The *semi-direct product* $S \rtimes_{\phi} T$ of S and T (with respect to the homomorphism ϕ) is the set $S \times T$ with multiplication of pairs defined by the rule

$$(s_1, t_1)(s_2, t_2) = (s_1\, {}^{t_1}s_2, t_1 t_2).$$

The direct product is actually a special case of the semi-direct product - the simplest case where the homomorphism ϕ maps every element of T to the identity endomorphism of S. Obviously the endomorphism semigroup of S always has identity. In the definition above there is nothing to say that if T is a monoid then the element 1_T must map under ϕ to this identity. In the following example this is not the case:

Example 1.18 Let S be the left zero semigroup on n elements $\{x_1, \ldots, x_n\}$ and let T be a monoid. Any map $f : S \to S$ is an endomorphism, since for $x, y \in S$ we have

$$(xy)f = xf = (xf)(yf).$$

We define g to be the endomorphism of S mapping the whole of S to x_n. Clearly g is idempotent, and if we define $\phi : T \to \text{End}(S)$ mapping T to g then ϕ is a homomorphism, and we may form the semi-direct product $S \rtimes_{\phi} T$.

However, for the purposes of this thesis, we will assume that any semi-direct products considered are 'well behaved', in the sense that if $\phi : T \to \text{End}(S)$ is a homomorphism between two monoids, then ϕ maps the identity to the identity.

It is well known that if $S = \langle A \,|\, R \rangle$ and $T = \langle B \,|\, Q \rangle$ are monoids then the (well behaved) semi-direct product $S \rtimes_{\phi} T$ is defined by the (monoid) presentation

$$\langle A, B \,|\, R, Q, ba = {}^{b}a b \ (a \in A, b \in B) \rangle$$

Let S and T be semigroups. We form the unrestricted direct product of $|T|$ copies of S as described in Section 3. Note that the unrestricted direct product S^T and the restricted direct product $S^{(T)}$ are identical provided T is finite and S contains an idempotent element. We will find it easiest to use the function descriptions of S^T and $S^{(T)}$ in what is to follow.

Given a function $f \in S^T$ and an element $t \in T$ we define ${}^t f$ to be the function mapping $x \in T$ to $(xt)f$ in S. It is obvious that ${}^t(fg) = {}^t f\, {}^t g$ and so each element t of T defines an endomorphism ϕ_t of S^T. The map ϕ from T to $\mathrm{End}(T)$ taking t to ϕ_t is a homomorphism, since

$$(x)(f\phi_{t_1 t_2}) = (x)\, {}^{t_1 t_2}f = (xt_1t_2)f = (xt_1)\, {}^{t_2}f = (x)\, {}^{t_2}({}^{t_1}f) = (x)(f\phi_{t_1}\phi_{t_2}),$$

and so we may form the semi-direct product $S^T \rtimes_\phi T$.

Definition 1.20 Let S and T be semigroups. The *unrestricted wreath product* $S\mathrm{Wr}T$ of S and T is the semi-direct product $S^T \rtimes_\phi T$ where ϕ is defined as above. In other words, $S\mathrm{Wr}T$ is the set of all pairs (f, t) where $f : T \to S$ and $t \in T$, with multiplication given by the rule

$$(f_1, t_1)(f_2, t_2) = (f_1\, {}^{t_1}f_2, t_1t_2).$$

As remarked before, if T is infinite and S is non-trivial, then S^T is uncountable, and consequently so is $S\mathrm{Wr}T$. However, as for the direct product, we may form a restricted version of the wreath product. Again, as in the direct product case, we need to fix a specific idempotent element e in S.

Definition 1.21 Let S and T be semigroups, and let e be a fixed idempotent element of S. The *(restricted) wreath product* of S and T (with respect to e) denoted $S\mathrm{wr}T$ (or $S_e\mathrm{wr}T$ to emphasize the chosen idempotent), is the subsemigroup of $S\mathrm{Wr}T$ generated by the set

$$X = \{(f, t) : f \in S^{(T)}, t \in T\}.$$

The semigroups S and T are respectively known as the *bottom* and *top* semigroups of the wreath product $S\mathrm{wr}T$.

This definition is equivalent to that found in [16]. We note that if S and T are countable, then $S\mathrm{wr}T$ is countably generated, and hence countable. We also observe that if $S\mathrm{wr}T$ is finitely generated by an infinite set X say, then by Theorem 1.4 a finite subset of X generates $S\mathrm{wr}T$. That is, although the wreath product may contain an infinite number of elements (f, t) where f does not have finite support (with respect to a fixed idempotent e), if $S\mathrm{wr}T$ is finitely generated then we may always choose a finite generating set $\{(f_i, t_i) : i \in I\}$ where each f_i has finite support (with respect to e). We will use this fact repeatedly later on. Provided S has an idempotent, we have that the restricted and unrestricted wreath products of S and T are equal if and only if either S is trivial or T is finite. Our definition allows us to define the wreath product $S_e\mathrm{wr}T$ with respect to any idempotent e of S, although if S is a monoid, we will generally choose $e = 1_S$ unless explicitly stated otherwise.

This definition of the wreath product differs from the well known definition for groups. In the group case we form the semi-direct product of $S^{(T)}$ and T, using the same action as for the unrestricted wreath product. Unfortunately endomorphisms in the image of T under the homomorphism $\phi : T \to \mathrm{End}(S^T)$ do not in general restrict to endomorphisms of $S^{(T)}$ as they do in the group case, as the following example illustrates:

Example 1.22 Let $T = G^0$ be an infinite group with a multiplicative zero adjoined, and let $S = \{1, s\}$ be the cyclic group of order two. Recall that by $\overline{s_0}$ we mean the function from T to S mapping 0 to s and the rest of T to the identity 1. Now $\overline{s_0}$ is clearly an element of $S^{(T)}$, but the function ${}^0\overline{s_0}$ maps x to $(x0)\overline{s_0}$, i.e. to s. Thus ${}^0\overline{s_0}$ is the constant map $T \to \{s\}$, and does not have finite support. Thus the endomorphism of S^T corresponding to $0 \in T$ does not restrict to an endomorphism of $S^{(T)}$.

It is because of examples such as this that we are forced to define the restricted wreath product in this less intuitive way. For groups, the elements of the restricted wreath product $S\mathrm{wr}T$ are simply pairs (f, t) where f is a function from T into S with finite support (with respect to the identity 1_S), and t is an element of T. The problem of describing elements in the wreath product of two monoids or semigroups is more complicated. See Chapter 5 for some general observations and simple examples. The simplest observation we can make is that while acting on a function f with an element of T need not preserve finite support, it does preserve finite image. Indeed, $(T)\,{}^tf = (Tt)f \subseteq (T)f$. Thus the function component of any pair (f, t) in the restricted wreath product of two semigroups necessarily has finite image.

The reason that finite support fails in Example 1.22 is that the element 0 maps infinitely many elements of T into 0 (by right multiplication). We can formalize this idea by writing bt^{-1} to denote the set $\{x \in T : xt = b\}$. In a group G, we usually think of bt^{-1} as an element of G rather than a subset of G consisting of a single element, although in monoids in general, bt^{-1} may be finite, infinite, or empty. In Example 1.22 we see that 00^{-1} is the whole of T. In general if f maps b to s then ${}^tf$ maps bt^{-1} to s. We have the following lemma:

Lemma 1.23 *Let f be a function from T into S with finite support (with respect to e). Let $t \in T$ be an element such that the sets bt^{-1} are finite for all elements b in the support of f. Then ${}^tf$ also has finite support.*

PROOF. Suppose that $x\,{}^tf \neq e$. Then $(xt)f \neq e$, and so $xt \in \mathrm{supp}(f)$. If we let $b = xt$, then $x \in bt^{-1}$ where $b \in \mathrm{supp}(f)$, and so by hypothesis there are only finitely many choices for x. Thus ${}^tf$ has finite support as required. ∎

Using this lemma we may prove the following theorem, which appears in [16, Chapter 10]. This tells us exactly when the function components of all pairs (f,t) in the wreath product $S\mathrm{wr}T$ have finite support. This is the case exactly when the semi-direct product $S_e^{(T)} \rtimes T$ exists and is equal to the wreath product $S_e\mathrm{wr}T$.

Theorem 1.24 *Let S and T be semigroups, S having idempotent element e. All elements (f,t) in $S_e\mathrm{wr}T$ have function component with finite support if and only if one of the following holds:*

(i) *The idempotent e is a left zero for S.*

(ii) *The sets bt^{-1} are all finite (or empty), for $b,t \in T$.*

PROOF. ($\Rightarrow$) Suppose that f has finite support for all $(f,t) \in S_e\mathrm{wr}T$, and that there exist $b,t \in T$ such that bt^{-1} is infinite. Let $s \in S$ be arbitrary. Let $\overline{s_b}$ be the function from T into S mapping b to s and the rest of T to e. Let $\overline{e}$ denote the function mapping the whole of T to e, and let

$$(f,t') = (\overline{e},t)(\overline{s_b},t).$$

By hypothesis, $f = \overline{e}\,{}^t\overline{s_b}$ has finite support. Now $(x)\,{}^t\overline{s_b} = s$ and so $xf = es$ for all $x \in bt^{-1}$, an infinite set. Thus we must have $es = e$, and since s was chosen arbitrarily, e is a left zero for S as required.

($\Leftarrow$) Suppose first that e is a left zero for S. By definition, any element (f,t) of $S_e\mathrm{wr}T$ can be written as a product

$$(f,t) = (f_1,t_1)(f_2,t_2)\dots(f_r,t_r)$$

where each f_i has finite support. Then we have $\mathrm{supp}(f) \subseteq \mathrm{supp}(f_1)$, because if $x \notin \mathrm{supp}(f_1)$ then $xf \in eS = e$. Alternatively, if we assume that all the sets bt^{-1} are finite, then, by Lemma 1.23, for any f with finite support and any $t \in T$ the function ${}^{t}f$ must have finite support. Now given an element (f,t) of $S_e\mathrm{wr}T$ we may write

$$(f,t) = (f_1,t_1)(f_2,t_2)\dots(f_r,t_r),$$

where each f_i has finite support, and so $f = f_1\, {}^{t_1}f_2 \dots {}^{t_1\dots t_{r-1}}f_r$ must have finite support as required. ■

We note that the infinite support in Example 1.22 was obtained by acting on a certain function with the zero element of T. Of course once 0 has occurred in the T component in a product of pairs $(f_1,t_1)\dots(f_r,t_r)$ we are stuck with it - no matter what the other values of t_i, we will always end up with 0 in the T component of the product. If we do not want t to be 0 in an element $(f,t) \in S_e\mathrm{wr}T$ then, as we will see in Corollary 1.29 below, f must have finite support. First we make the following definitions.

Definition 1.25 Let T be a semigroup. An element $t \in T$ is said to be *(right) injective* if whenever $at = bt$ for $a, b \in T$ we have $a = b$.

Obviously any group or cancellative semigroup consists entirely of injective elements. The units in any monoid are also injective. Injective elements satisfy the following:

Lemma 1.26 *Let f be a function from T into S with finite support, and let t be an injective element. Then ${}^{t}f$ also has finite support.*

PROOF. This follows from Lemma 1.23, observing that if t is injective, then the set bt^{-1} contains at most one element. ■

Definition 1.27 Let T be a semigroup. We define the (right) ancestry of t, $A\mathcal{R}_t$ to be the set $\{t' \in T : t \in t'T\}$.

If t is injective and $t \in t'T$, then $at' = bt'$ implies $at = bt$ which implies $a = b$, and so t' is also injective. Thus the ancestry of an injective element contains only injective elements. Also, if $t_1t_2 = t$, and $x \in bt_1^{-1}$, then $xt_1 = b$ and so $xt = xt_1t_2 = bt_2$. Thus

$$bt_1^{-1} \subseteq (bt_2)t^{-1}.$$

It follows that if, the sets bt^{-1} are finite for all $b \in T$ then so are the sets bq^{-1}, for all $b \in T, q \in A\mathcal{R}_t$.

Corollary 1.28 *Let S and T be semigroups, S having idempotent element e. Fix an element $t \in T$. All elements (f, t) in $S_e\mathrm{wr}T$ have function component with finite support if and only if one of the following holds:*

(i) *The idempotent e is a left zero for S.*

(ii) *The sets bq^{-1} are all finite (or empty), for $b \in T, q \in A\mathcal{R}_t$.*

Proof. The proof we give is very similar to that of Theorem 1.24.

($\Rightarrow$) Let t be a fixed element of T and suppose that f has finite support for all $(f,t) \in S_e\mathrm{wr}T$. Suppose also that there exist $b \in T, t_1 \in A\mathcal{R}_t$ such that bt_1^{-1} is infinite. Let t_2 be such that $t_1t_2 = t$. Let $s \in S$ be arbitrary. Let $\overline{s_b}$ be the function from T into S mapping b to s and the rest of T to e. Let $\overline{e}$ denote the function mapping the whole of T to e. Then we have

$$(\overline{e}, t_1)(\overline{s_b}, t_2) = (f, t),$$

where $f = \overline{e}\,{}^{t_1}\overline{s_b}$ has finite support by hypothesis. Now $(x)\,{}^{t_1}\overline{s_b} = s$ and so $xf = es$ for all $x \in bt_1^{-1}$, an infinite set. Thus we must have $es = e$, and since s was chosen arbitrarily, e is a left zero for S as required.

($\Leftarrow$) Let t be a fixed element of T. Suppose first that e is a left zero for S. By definition, any element (f, t) of $S_e\mathrm{wr}T$ can be written as a product

$$(f, t) = (f_1, t_1)(f_2, t_2) \ldots (f_r, t_r)$$

where each f_i has finite support. Then we have $\text{supp}(f) \subseteq \text{supp}(f_1)$, because if $x \notin \text{supp}(f_1)$ then $xf \in eS = e$. Alternatively, assume that the sets bq^{-1} are finite, for all $b \in B, q \in A\mathcal{R}_t$. By Lemma 1.23, for any f with finite support and any $q \in A\mathcal{R}_t$ the function ${}^q f$ must have finite support. Now given an element (f,t) of $S_e\text{wr}T$ we may write

$$(f,t) = (f_1,t_1)(f_2,t_2)\ldots(f_r,t_r),$$

where each f_i has finite support, and $t_1 \ldots t_j \in A\mathcal{R}_t$ for $1 \leqslant j < r$. Thus $f = f_1\, {}^{t_1}f_2 \ldots {}^{t_1\ldots t_{r-1}}f_r$ is a product of functions with finite support, and so has finite support as required. ∎

Corollary 1.29 *Let T be a monoid, and S a semigroup with fixed idempotent element e. The pair $(f, 1_T)$ is an element of $S\text{wr}T$ if and only if f has finite support.*

PROOF. If f has finite support then $(f, 1_T)$ is an element of the infinite generating set we used to define $S\text{wr}T$. The converse follows from Corollary 1.28, since 1_T is clearly injective. ∎

We note that we could exchange 1_T for any injective element t of T and the above corollary would still hold.

As a final remark, we note that the definition of the unrestricted wreath product $S\text{Wr}T$ of two semigroups is not left-right symmetric. We may consider the mapping $f \in S^T$ as acting on the left, and take the natural right action $(f,t) \mapsto f^t$ of T on S^T. Taking the corresponding semi-direct product we obtain the *left unrestricted wreath product* $S\text{Wr}_l T$ of S by T. By considering the mappings $f \in S^T$ with finite support, we obtain the *left (restricted) wreath product* $S\text{wr}_l T$ in the same way as before. In general $S\text{Wr}T \ncong S\text{Wr}_l T$ and $S\text{wr}T \ncong S\text{wr}_l T$, although they are isomorphic when T is a group. The various theorems that we will prove for the (usual) wreath product will have left-right analogues, and we will usually state these without proof.

Chapter 2

Generators and Relations of Wreath Products of Monoids

1 Introduction

In this chapter we investigate generating sets and relations for wreath products of monoids, and in particular give necessary and sufficient conditions for the wreath product to be finitely generated and finitely presented. We start by looking at the special case of two groups. The following theorem is well known, we prove it here for completeness.

Theorem 2.1 *Let G and H be groups. Then the wreath product $G\mathrm{wr}H$ is finitely generated if and only if both G and H are finitely generated.*

PROOF. Suppose that the groups G and H are generated by finite sets X and Y respectively. We will show that the finite set

$$Z = \{(\overline{x_1}, 1) : x \in X\} \cup \{(\overline{1}, y) : y \in Y\}$$

generates $G\mathrm{wr}H$. Recall that $\overline{x_1}$ is the notation we use for the function from H to G mapping 1_H to x and everything else to 1_G. With the usual action of H on

$G^{(H)}$ we have

$$^{h^{-1}}\overline{x_1} = \overline{x_h} \quad (h \in H, x \in X)$$

and so

$$(\overline{1}, h^{-1})(\overline{x_1}, 1)(\overline{1}, h) = (\overline{x_h}, 1).$$

Obviously $(\overline{x_h}, 1)(\overline{x'_h}, 1) = (\overline{xx'_h}, 1)$, and so we may build up any element $(f, 1)$ from the generators in Z (where $f \in G^{(H)}$). Finally, we may multiply $(f, 1)$ by elements of $\{(\overline{1}, y) : y \in Y\}$ to give the required element of $G\mathrm{wr}H$.

Conversely suppose that $G\mathrm{wr}H$ is finitely generated. Then H, as a homomorphic image of $G\mathrm{wr}H$ (under the mapping $(f, h) \mapsto h$), is also finitely generated. Now, suppose that $G\mathrm{wr}H$ is generated by the set $\{(f_i, h_i) : i \in I\}$ where $f_i \in G^{(H)}$ and $h_i \in H$. We claim that G must be generated by the set $\bigcup_{i \in I}(H)f_i$ (this set is finite since the image of each f_i must be finite). Indeed, given $g \in G$ we simply write

$$(\overline{g_1}, 1) = (f_{i_1}, h_{i_1}) \ldots (f_{i_r}, h_{i_r})$$

and note that

$$g = (1)\overline{g} = (1)f_{i_1}\, {}^{h_{i_1}}f_{i_2} \ldots {}^{h_{i_1} \ldots h_{i_{r-1}}}f_{i_r} \in \left\langle \bigcup_{i \in I}(H)f_i \right\rangle$$

as required. ∎

The same theorem fails for monoids though, as the following example shows:

Example 2.2 Let $A \simeq B \simeq \mathbb{N}$ be the monoid of natural numbers (including 0) under addition. Then although A and B are finitely generated, $A\mathrm{wr}B$ is not.

Proof. We take a finite subset $X = \{(f_i, b_i) : i \in I\}$ of $A\mathrm{wr}B$. We will show that $\langle X \rangle \neq A\mathrm{wr}B$. Now, as shown in Chapter 1, Theorem 1.24, $A\mathrm{wr}B = A^{(B)} \rtimes B$ if and only if the sets bc^{-1} $(b, c \in B)$ are all finite. This is clearly the case when

$B = \mathbb{N}$ because the sets

$$bc^{-1} = \begin{cases} \{(b-c)\} & \text{if } b \geqslant c \\ \varnothing & \text{if } b < c \end{cases}$$

have size 0 or 1. So if $(f, b) \in A\mathrm{wr}B$ then f has finite support. In particular, each f_i has finite support, and we may choose N such that

$$N > \max\{n \in \mathbb{N} : n \in \cup_{i \in I} \operatorname{supp}(f_i)\}$$

By choice of N we have $(p)f_i = 0$ for all $p \geqslant N$ and all $i \in I$. Thus

$$(N)\,{}^{b}f_i = (N+b)f_i = 0$$

for all $b \in B$, $i \in I$. Now, given $(f, b) \in \langle X \rangle$ write

$$(f, b) = (f_1, b_1)(f_2, b_2)\ldots(f_r, b_r)$$

where $(f_j, b_j) \in X$ for $1 \leqslant j \leqslant r$. Then

$$f = f_1\, {}^{b_1}f_2 \ldots {}^{b_1+\ldots+b_{r-1}}f_r$$

and so

$$(N)f = (N)f_1(N+b_1)f_2\ldots(N+b_1+\ldots+b_{r-1})f_r = 0.$$

Therefore $\langle X \rangle$ cannot be the whole of $A\mathrm{wr}B$, since elements such as $(\overline{2_N}, 0)$ cannot be generated. ∎

The question of finite presentability of the wreath product of two groups was answered by Baumslag in [3]. His result below is simply stated, although the proof is quite technical.

Theorem 2.3 (Baumslag) *Let G and H be groups. Then the wreath product $G\mathrm{wr}H$ is finitely presented if and only if either G is trivial and H is finitely presented, or H is finite and G is finitely presented.* ∎

In this chapter we give analogous results to Theorems 2.1 and 2.3 for the wreath product of two monoids A and B. In the process of doing this we construct a generating set for $A\mathrm{wr}B$, and in the case where the sets $b_1b_2^{-1}$ are all finite ($b_1, b_2 \in B$) we give a (usually infinite) presentation. Presentations in the case where not all sets $b_1b_2^{-1}$ are finite (for example, if B is infinite with a multiplicative zero element) are more complicated, see Chapter 5 for more details.

2 Generators

The main purpose of this section is to prove the following:

Theorem 2.4 *Let A and B be monoids, and let G be the group of units of B. Then the wreath product $A\mathrm{wr}B$ is finitely generated if and only if both A and B are finitely generated, and either A is trivial, or $B = VG$ for some finite subset V of B.*

Before proving Theorem 2.4 we look at some examples.

Example 2.5 If B is a group, then $B = G = \{1_B\}G$, and so $A\mathrm{wr}B$ is finitely generated if and only if both A and B are finitely generated. In particular, if A is also a group then we simply reduce to Theorem 2.1 for groups.

Example 2.6 Let T be any infinite semigroup, and let $M = T^1$ be the monoid obtained by adjoining an identity. Then M has trivial group of units, and so there cannot exist a finite set V such that $M = V\{1_M\}$. Thus $S\mathrm{wr}M$ is not finitely generated for any non-trivial semigroup S.

Example 2.7 Let G and H be infinite finitely generated isomorphic groups and let $\phi : G \to H$ be an isomorphism. Let $T_1 = G \cup H$. Then T_1 is a monoid with multiplication $*$ if we define

$$g_1 * g_2 = g_1 g_2 \in G \qquad g * h = (g\phi)h \in H$$
$$h_1 * h_2 = h_1 h_2 \in H \qquad h * g = h(g\phi) \in H$$

This is a special case of the strong semi-lattice of groups construction, see [12, Chapter 4] for details. The element 1_G is an identity for T_1, and the group of units is simply G. It is easy to see that $T_1 = \{1_G, 1_H\}G$ and so $S\mathrm{wr}T_1$ is finitely generated for any finitely generated monoid S.

Let $\psi : G \to H$ be the trivial homomorphism $g \mapsto 1_H$. Construct a monoid $T_2 = G \cup H$ as before, using the map ψ instead of ϕ to define multiplication. Again T_2 has group of units G, but $VG \subseteq G \cup V$ for any subset V of T_2, and so cannot be the whole of T_2 if V is finite. Thus $S\mathrm{wr}T_2$ is not finitely generated for any non-trivial monoid S.

Now we construct an explicit (possibly infinite) generating set for the wreath product of two arbitrary monoids.

Theorem 2.8 *Let A and B be monoids generated by sets X and Y respectively, and let $V \subseteq B$ be such that $B = VG$ where G is the group of units of B. Then the wreath product $A\mathrm{wr}B$ is generated by the set*

$$\big\{(\overline{x_v}, 1_B) : x \in X, v \in V\big\} \cup \big\{(\overline{1}, y) : y \in Y\big\}.$$

PROOF. First recall that, by definition, every element of $A\mathrm{wr}B$ is a product of elements of the form (f, b), where $\mathrm{supp}(f)$ is finite. Consider now such an element (f, b). Write $b = y_1 \dots y_r$ $(y_i \in Y)$ and also note that

$$f = \prod_{z \in \mathrm{supp}(f)} \overline{(zf)_z},$$

giving

$$(f, b) = \Big(\prod_{z \in \mathrm{supp}(f)} (\overline{(zf)_z}, 1_B)\Big)\Big(\prod_{i=1}^{r} (\overline{1}, y_i)\Big).$$

Now fix $z \in \text{supp}(f)$ and write it as $z = vg$ ($v \in V$, $g \in G$). Also write $zf = x_1 \dots x_s$ ($x_i \in X$). Now we have

$$\big(\overline{(zf)_z}, 1_B\big) = \big(\overline{1}, g^{-1}\big)\big(\overline{(zf)_v}, 1_B\big)\big(\overline{1}, g\big) = \big(\overline{1}, g^{-1}\big)\Big(\prod_{i=1}^{s}\big(\overline{(x_i)_v}, 1_B\big)\Big)\big(\overline{1}, g\big).$$

Finally, decomposing g and g^{-1} into products of generators from Y completes the proof. ∎

Note that if X, Y and V are finite in the above theorem, then the generating set constructed for $A\text{wr}B$ is finite, and so we have proved one half of Theorem 2.4. Now we do the opposite of Theorem 2.8, we take an arbitrary generating set for $A\text{wr}B$ and use it to construct generating sets for A and B.

Theorem 2.9 *Let A and B be monoids. If $A\text{wr}B$ is generated by a set Z then A is generated by the set*

$$X = \big\{ {}^{b}f : b \in B \text{ and } (\exists c \in B)((f, c) \in Z)\big\}$$

and B is generated by the set

$$Y = \big\{ b \in B : (\exists f \in A^B)((f, b) \in Z)\big\}.$$

In particular, if $A\text{wr}B$ is finitely generated then both A and B are also finitely generated.

Proof. Let $a \in A$, $b, c \in B$ be arbitrary. Write $(\overline{a_c}, b)$ as a product of generators from Z:

$$(\overline{a_c}, b) = (f_1, b_1)(f_2, b_2)\dots(f_r, b_r) = (f_1\, {}^{b_1}f_2 \dots {}^{b_1 \dots b_{r-1}}f_r, b_1 b_2 \dots b_r).$$

Then

$$\begin{aligned} a &= (c)\overline{a_c} = ((c)f_1)((cb_1)f_2)\dots((cb_1 \dots b_{r-1})f_r) \in \langle X \rangle, \\ b &= b_1 b_2 \dots b_r \in \langle Y \rangle. \end{aligned}$$

If Z is finite then it is obvious that Y is finite. In fact X is finite too, since as noted in Chapter 1, Section 4, the function component f in an element $(f, c) \in A\mathrm{wr}B$ may sometimes have infinite support, but always has finite image. That is, Bf is always finite. Hence if $A\mathrm{wr}B$ is finitely generated, so are both A and B. ■

We note here that Theorem 2.9 also holds for arbitrary semigroups. Nowhere in the proof do we use the fact that either S or T has an identity.

Next we prove the following technical lemma, which is needed in the proof of Theorem 2.4.

Lemma 2.10 *Let A and B be monoids, and assume that A is non-trivial. If $A\mathrm{wr}B$ is finitely generated then there exists a finite set $V \subseteq B$ such that for every $b \in B$ there exists a right invertible element $t_b \in B$ such that $bt_b \in V$.*

PROOF. Since $A\mathrm{wr}B$ is generated by $A^{(B)} \times B$ and since it is finitely generated, it follows using Theorem 1.4 that there exists a finite generating set Z for $A\mathrm{wr}B$ which is contained in $A^{(B)} \times B$. Define

$$V = \bigcup_{(f,b)\in Z} \mathrm{supp}(f);$$

this is finite since Z and all $\mathrm{supp}(f)$ are finite. Fix an arbitrary $b \in B$. For any $a \in A \setminus \{1_A\}$ write $(\overline{a_b}, 1_B)$ as a product of generators from Z:

$$(\overline{a_b}, 1_B) = (f_1, b_1)(f_2, b_2)\dots(f_r, b_r).$$

Equating the components we obtain

$$\overline{a_b} = f_1\,{}^{b_1}f_2 \dots {}^{b_1\dots b_{r-1}}f_r \tag{2.1}$$

$$1_B = b_1 b_2 \dots b_r. \tag{2.2}$$

Evaluating (2.1) at b we obtain

$$1_A \neq a = (b)\overline{a_b} = ((b)f_1)((bb_1)f_2)\dots((bb_1\dots b_{r-1})f_r).$$

Clearly there must exist $i \in \{1, \ldots, r\}$ such that $(bb_1 \ldots b_{i-1})f_i \neq 1_A$; let $t_b = b_1 \ldots b_{i-1}$ (and $t_b = 1_B$ if $i = 1$). Then we have $bt_b \in \text{supp}(f_i) \subseteq V$, while from (2.2) we have $t_b b_i \ldots b_r = 1_B$. ∎

Corollary 2.11 *Let A and B be monoids, with A non-trivial. If $A\text{wr}B$ is finitely generated then:*

(i) *B has only finitely many $\mathcal{R}$-classes.*

(ii) *Every right invertible element of B is also left invertible (and vice-versa).*

(iii) *If G is the group of units of B then $B \setminus G$ is an ideal in B.*

PROOF. (i) By Lemma 2.10, every element of B is $\mathcal{R}$-related to an element of the finite set V.

(ii) Let $b \in B$ be right invertible, and let c be a right inverse for b. Obviously c is left invertible. Since B has only a finite number of $\mathcal{R}$-classes there must exist integers $k > j \geqslant 1$ such that $c^k \mathcal{R} c^j$, i.e. $c^k y = c^j$ for some $y \in B$. Then $c^{k-j} y = b^j c^k y = b^j c^j = 1_B$ and so c is right invertible, and hence invertible. Thus b, as a left inverse of an invertible element, must also be invertible.

(iii) This is an immediate consequence of (ii). ∎

We can now prove the main theorem of this section, restated below:

Theorem 2.4 *Let A and B be monoids, and let G be the group of units of B. Then the wreath product $A\text{wr}B$ is finitely generated if and only if both A and B are finitely generated, and either A is trivial, or $B = VG$ for some finite subset V of B.*

Proof. If A is trivial then $A\mathrm{wr}B \cong B$ and there is nothing to prove. So we concentrate on the case where A is non-trivial.

($\Leftarrow$) By Theorem 2.8, if all X, Y and V are finite, then so is the generating set Z for $A\mathrm{wr}B$.

($\Rightarrow$) Assume that $A\mathrm{wr}B$ is finitely generated. By Theorem 2.9 we have that both A and B are finitely generated. Now choose a finite set $V \subseteq B$ as in Lemma 2.10. For $b \in B$ we have $bt_b = v \in V$ where t_b is right invertible. By Corollary 2.11 (ii), t_b must be invertible, so that $b = vt_b^{-1} \in VG$ as required. ∎

3 Finite Presentability

This section is dedicated to the proof of the following result, analogous to Theorem 2.3.

Theorem 2.12 *The wreath product $A\mathrm{wr}B$ of two monoids A and B is finitely presented if and only if either A is finitely presented and B is finite, or A is trivial and B is finitely presented.*

It turns out that the necessary and sufficient conditions are the same as those found by Baumslag for groups, although we need to use the 'non-group' property $B = VG$ from Theorem 2.4 to prove it. To start with, we give the presentation for $A\mathrm{wr}B$ given by Howie and Ruskuc in [13].

Theorem 2.13 *Let A and B be monoids defined by the presentations $\langle X \mid R \rangle$ and $\langle Y \mid Q \rangle$ respectively. Let X_b be copies of the alphabet X indexed by $b \in B$ and let R_b be the corresponding copies of R. Suppose that the sets bc^{-1} are finite* *for all $b, c \in B$. Then the presentation*

$$\mathcal{P} = \langle X_b \ (b \in B),\, Y \mid R_b \ (b \in B),\, Q,\, yx_b = (\prod_{c \in by^{-1}} x_c)y \quad (y \in Y,\, b \in B),$$
$$x_d x'_e = x'_e x_d \quad (d, e \in B, d \neq e,\ x, x' \in X)\rangle$$

defines the monoid Awr$B$*. In particular, if B is finite and A is finitely presented then A*wr$B$ *is finitely presented.*

We note that the condition that the sets bc^{-1} are all finite is required (a fact which was overlooked in [13]) because otherwise we would have words of infinite length in our presentation. For completeness, we give the proof given in [13].

Proof. Let ϕ be the mapping from $((\bigcup_{b \in B} X_b) \cup Y)^*$ onto $A\mathrm{wr}B$ defined by

$$x_b\phi = (\overline{x}_b, 1_B),\ x \in X,\ b \in B,$$
$$y\phi = (\overline{1}, y),\ y \in Y.$$

Note that the function ${}^{y}\overline{x_b}$ maps the set by^{-1} to x, and everything else to 1_A, and so

$$(\overline{1}, y)(\overline{x_b}, 1_B) = (\prod_{c \in by^{-1}} \overline{x_c}, 1_B)(\overline{1}, y).$$

Therefore the relations $yx_b = (\prod_{c \in by^{-1}} x_c)y$ map under ϕ to relations in $A\mathrm{wr}B$. The other relations in the presentation $\mathcal{P}$ clearly hold in $A\mathrm{wr}B$ and so ϕ induces a homomorphism from the monoid M defined by $\mathcal{P}$ to $A\mathrm{wr}B$, in fact an epimorphism since ϕ is onto by Theorem 2.8. To prove that the presentation $\mathcal{P}$ does indeed define $A\mathrm{wr}B$ it remains only to show that ϕ is injective.

Let w be any word representing an element of M. It is easy to use relations from $\mathcal{P}$ to show that there exist words $w(b)$ in X^* ($b \in B$) and w' in Y^* such that

$$w = (\prod_{b \in B} (w(b))_b) w'$$

in M. (Here, if $z \in X^*$ then z_b is the corresponding word in X_b^*.) Now for each w in X^* and each c in B we have

$$c\overline{w}_b = \begin{cases} w & \text{if } c = b \\ 1 & \text{otherwise.} \end{cases}$$

Hence

$$c(\prod_{b\in B}(\overline{w(b)})_b) = \prod_{b\in B} c(\overline{w(b)})_b = w(c) \tag{2.3}$$

for all c in B.

For any two words u, v in $((\bigcup_{b\in B} X_b)\bigcup Y)^*$ we have

$$\begin{aligned} u\phi = v\phi &\implies ((\prod_{b\in B}(u(b))_b)u')\phi = ((\prod_{b\in B}(v(b))_b)v')\phi \\ &\implies (\prod_{b\in B}((u(b))_b\phi))(u'\phi) = (\prod_{b\in B}((v(b))_b\phi))(v'\phi) \\ &\implies (\prod_{b\in B}(\overline{u(b)})_b, 1_B)(\overline{1}, u') = (\prod_{b\in B}(\overline{v(b)})_b, 1_B)(\overline{1}, v') \\ &\implies (\prod_{b\in B}(\overline{u(b)})_b, u') = (\prod_{b\in B}(\overline{v(b)})_b, v'). \end{aligned}$$

Now from the equality of the first components we deduce using (2.3) that $u(c) = v(c)$ in A for every c in B, and the equality of the second components gives $u' = v'$ in B. The relations R_b and Q in $\mathcal{P}$ then give us that $u = v$ in M, and hence ϕ is injective.

Now, if B is finite then the conditions that bc^{-1} are finite are trivially satisfied, and so the presentation given above defines $A\mathrm{wr}B$. If further A is finitely presented then X, Y, R and Q may all be chosen to be finite and so $A\mathrm{wr}B$ is finitely presented as required. ∎

If A is trivial and B is finitely presented then $A\mathrm{wr}B \simeq B$ is also finitely presented. This fact along with Theorem 2.13 prove the indirect half of Theorem 2.12 The proof of the direct half is technically more complicated and we adopt the following strategy. First we dispose of the case where A is trivial. Indeed, then $B \simeq A\mathrm{wr}B$ and so B is finitely presented. Then we consider the case where A is non-trivial. We prove that $A\mathrm{wr}B$ finitely presented implies that $A\mathrm{wr}G$ is finitely presented, where G is the group of units of B; see Proposition 2.14. Then we prove that $A\mathrm{wr}G$ finitely presented implies that A is finitely presented (Proposition 2.16). Next, basing our approach on that of Baumslag [3], we prove that $A\mathrm{wr}G$ finitely presented implies that G is finite (Proposition 2.17). Finally we invoke our finite generation result (Theorem 2.4), which gives us that $B = VG$ where $V \subseteq B$ is finite, implying that B is finite.

Proposition 2.14 *Let A be a non-trivial monoid, let B be any monoid, and let G be the group of units of B. If $A\mathrm{wr}B$ is finitely presented, then so is $A\mathrm{wr}G$.*

PROOF. Consider the submonoid

$$M = \{(f, b) \in A\mathrm{wr}B : b \in G\}$$

of $A\mathrm{wr}B$. By Corollary 2.11 (iii), $B \setminus G$ is an ideal of B, so that $(A\mathrm{wr}B) \setminus M$ is an ideal of $A\mathrm{wr}B$. This implies that M is finitely presented, using Theorem 1.6. Now define a mapping $\phi : M \to A\mathrm{wr}G$ by

$$(f, b)\phi = (f\restriction_G, b).$$

Note that for any $f \in A^B$ and any $b \in G$ we have ${}^b(f\restriction_G) = ({}^bf)\restriction_G$ since G is a group. Therefore, for $(f_1, b_1), (f_2, b_2) \in M$ we have

$$\begin{aligned} ((f_1, b_1)(f_2, b_2))\phi &= (f_1\, {}^{b_1}f_2, b_1b_2)\phi = ((f_1\, {}^{b_1}f_2)\restriction_G, b_1b_2) = ((f_1\restriction_G)\, {}^{b_1}(f_2\restriction_G), b_1b_2) \\ &= (f_1\restriction_G, b_1)(f_2\restriction_G, b_2) = (f_1, b_1)\phi(f_2, b_2)\phi, \end{aligned}$$

so that ϕ is an epimorphism. It follows that $A\mathrm{wr}G$ is the quotient of the finitely presented monoid M by the congruence $\rho = \ker\phi$, and so the proof is complete once we establish the following:

Lemma 2.15 *Given the conditions in Proposition 2.14 then the relation $\rho = \ker \phi$ is a finitely generated congruence on M.*

PROOF. By Theorem 2.4, we can choose finite sets X, Y and V so that $A = \langle X \rangle$ and $B = \langle Y \rangle = VG$. We show that the congruence σ of M generated by the finite set

$$\left\{ \left((\overline{x_v}, 1_B), (\overline{1}, 1_B) \right) : x \in X, v \in V \setminus G \right\}$$

is equal to ρ. Obviously $\sigma \subseteq \rho$ since $\overline{x_v}$ and $\overline{1}$ are identical on G for $v \in V \setminus G$. Now consider $\left((f,b), (g,c) \right) \in \rho$; this means that $f \restriction_G = g \restriction_G$ and $b = c$. Since $(f,b) \in M \subseteq A\mathrm{wr}B$, it follows that we can write $(f,b) = (f_1,b_1)\ldots(f_r,b_r)$, where each f_i has finite support. It follows that

$$f = f_1 \, {}^{b_1}f_2 \ldots {}^{b_1 \ldots b_{r-1}}f_r \quad \text{and} \quad b = b_1 \ldots b_r.$$

Since $b \in G$ and $B \setminus G$ is an ideal (Corollary 2.11) we must have $b_i \in G$ ($i = 1, \ldots, r$). As noted in Lemma 1.26 we know that if g is invertible, then $|\operatorname{supp}({}^g f_i)|$ is finite, since invertible elements are obviously injective. (In fact it is not difficult to show that $|\operatorname{supp}({}^g f_i)| = |\operatorname{supp}(f_i)|$. Some similar results are proved in Chapter 4.) Thus ${}^{b_1 \ldots b_{i-1}}f_i$ has finite support for $1 \leqslant i \leqslant r$, and consequently f must also have finite support. From the definition of σ we have

$$(\overline{x_v}, 1_B) \ \sigma \ (\overline{1}, 1_B) \quad (x \in X, b \in V \setminus G).$$

Conjugating by $(\overline{1}, g)$ ($g \in G$) we obtain

$$\begin{aligned} (\overline{x_{vg}}, 1_B) &= (\overline{1}, g)^{-1}(\overline{x_v}, 1_B)(\overline{1}, g) \ \sigma \ (\overline{1}, g)^{-1}(\overline{1}, 1_B)(\overline{1}, g) \\ &= (\overline{1}, 1_B) \quad (x \in X, v \in V \setminus G, g \in G). \end{aligned}$$

Since $VG = B$ this is equivalent to

$$(\overline{x_b}, 1_B) \ \sigma \ (\overline{1}, 1_B) \quad (x \in X, b \in B \setminus G).$$

From $A = \langle X \rangle$ it now follows that

$$(\overline{a_b}, 1_B) \ \sigma \ (\overline{1}, 1_B) \ \ (a \in A, b \in B \setminus G).$$

Multiplying the above for $b \in \mathrm{supp}(f) \setminus G$ and $a = bf$ yields

$$(f', 1_B) \ \sigma \ (\overline{1}, 1_B) \tag{2.4}$$

where

$$zf' = \begin{cases} zf & \text{if } z \in B \setminus G \\ 1_A & \text{otherwise.} \end{cases}$$

Define $f'' : B \to A$ by

$$zf'' = \begin{cases} zf & \text{if } z \in G \\ 1_A & \text{otherwise,} \end{cases}$$

and note that $f = f'f''$, so that multiplying (2.4) by (f'', b) gives

$$(f, b) \ \sigma \ (f'', b).$$

Similarly, defining functions g' and g'' from B to A by

$$zg' = \begin{cases} zg & \text{if } z \in B \setminus G \\ 1_A & \text{otherwise,} \end{cases} \quad \text{and} \quad zg'' = \begin{cases} zg & \text{if } z \in G \\ 1_A & \text{otherwise,} \end{cases}$$

and repeating the above argument we get

$$(g, c) \ \sigma \ (g'', c).$$

Thus

$$(f, b) \ \sigma \ (f'', b) = (g'', c) \ \sigma \ (g, c)$$

and $\rho \subseteq \sigma$ as required. ∎

If A is trivial then the above result fails because there exists a finitely presented monoid with a non finitely presented group of units; see [24].

Proposition 2.16 *Let A be a monoid, and let G be a group. If* $A\mathrm{wr}G$ *is finitely presented then A is also finitely presented.*

PROOF. If $A\mathrm{wr}G$ is finitely presented then it is also finitely generated, and so both A and G are finitely generated by Theorem 2.4. Choose presentations $\langle X \mid R \rangle$ and $\langle Y \mid Q \rangle$ for A and G respectively with X and Y finite (and R and Q possibly infinite). By [13, Corollary 2.3] (see also the remarks at the end of Section 2) $A\mathrm{wr}G$ is defined by

$$\langle X,\, Y \mid R,\, Q,\, (g^{-1}xg)x' = x'(g^{-1}xg)\ (1 \neq g \in G,\, x, x' \in X) \rangle.$$

The finite presentability of $A\mathrm{wr}G$ implies that a finite subset of the above relations suffices to define $A\mathrm{wr}G$. In particular, there is a finite subset $R' \subseteq R$ such that

$$\langle X,\, Y \mid R',\, Q,\, (g^{-1}xg)x' = x'(g^{-1}xg)\ (1 \neq g \in G,\, x, x' \in X) \rangle \tag{2.5}$$

defines $A\mathrm{wr}G$, and the submonoid generated by X is isomorphic to A. On the other hand, again by [13, Corollary 2.3], (2.5) defines the wreath product of the monoid A' defined by $\langle X \mid R' \rangle$ and G, and the submonoid generated by X is isomorphic to A'. We conclude that $A \simeq A'$ is finitely presented. ∎

Proposition 2.17 *Let A be a non-trivial monoid, and let G be a group. If* $A\mathrm{wr}G$ *is finitely presented then G is finite.*

PROOF. Since $A\mathrm{wr}G$ is finitely presented it follows that both A and G are finitely generated, by Theorem 2.4. Let $\langle X \mid R \rangle$ and $\langle Y \mid Q \rangle$ be presentations for A and G respectively, with X and Y finite. By [13, Corollary 2.3], the wreath product $A\mathrm{wr}G$ is defined by

$$\big\langle X,\, Y \mid R,\, Q,\, g^{-1}xgx' = x'g^{-1}xg\ (x, x' \in X,\, g \in G \setminus \{1\}) \big\rangle. \tag{2.6}$$

Now, $A\mathrm{wr}G$ is finitely presented, so there exist finite subsets $R_0 \subseteq R$, $Q_0 \subseteq Q$ and $G_0 \subseteq G \setminus \{1\}$ such that the presentation

$$\langle X,\, Y \mid R_0,\, Q_0,\, g^{-1}xgx' = x'g^{-1}xg \ (x, x' \in X,\, g \in G_0)\rangle \qquad (2.7)$$

defines $A\mathrm{wr}G$. We now aim to show that this presentation cannot define $A\mathrm{wr}G$ unless G is finite. To this end we will first modify presentation (2.7) by adding some redundant generators and relations. However, we do this in such a way that when G is infinite we can find a relation which is not a consequence of the modified presentation, but holds in (2.6). First we note that we can add redundant relations $R \setminus R_0$ and $Q \setminus Q_0$ to (2.7). Next we define a set

$$T = \{(h, k) \in G \times G : hk^{-1} \in G_0\}.$$

By conjugating the relations $g^{-1}xgx' = x'g^{-1}xg$ from (2.7) by elements of G we see that each of the relations

$$h^{-1}xhk^{-1}x'k = k^{-1}x'kh^{-1}xh \ \ (x, x' \in X, (h, k) \in T)$$

is a consequence of (2.7), and so can be added to give the presentation

$$\langle X,\, Y \mid R,\, Q,\, h^{-1}xhk^{-1}x'k = k^{-1}x'kh^{-1}xh \ (x, x' \in X,\, (h, k) \in T)\rangle \qquad (2.8)$$

for $A\mathrm{wr}G$. For each $g \in G$ introduce a new alphabet $X_g = \{x_g : x \in X\}$ in 1-1 correspondence with X, and let R_g be the corresponding copy of R. Now we add redundant generators

$$x_g = g^{-1}xg \ (x \in X, g \in G) \qquad (2.9)$$

to (2.8). With this it is clear that the relations R_g $(g \in G)$ are consequences of R and Q and so we add them to (2.8) as well. From (2.9) we have $x_1 = x$ $(x \in X)$, and so $R = R_1$, and we obtain the presentation

$$\begin{aligned}\langle X_g \ (g \in G),\, Y \mid R_g \ (g \in G),\, Q,\, x_h x'_k = x'_k x_h \ (x, x' \in X_1,\, (h, k) \in T),\\ x_g = g^{-1}x_1 g \ (x \in X, g \in G)\rangle\end{aligned} \qquad (2.10)$$

for $A\mathrm{wr}G$. Next we note that for $l, g \in G$ and $x \in X$ we have

$$l^{-1}x_g l = l^{-1}g^{-1}x_1 gl = (gl)^{-1}x_1 gl = x_{gl}.$$

In particular, conjugating the relations from R_g by $l \in G$ gives R_{gl}. Similarly, conjugating the relations $x_h x'_k = x'_k x_h$ by $l \in G$ gives $x_{hl}x'_{kl} = x'_{kl}x_{hl}$. Note that if $(h,k) \in T$ then $(hl)(kl)^{-1} = hk^{-1} \in G_0$, so that $(hl, kl) \in T$, and so $x_{hl}x'_{kl} = x'_{kl}x_{hl}$ is also one of the relations from (2.10). We conclude that $A\mathrm{wr}G$ is a semidirect product of the monoid C by G, where C is defined by the presentation

$$\langle X_g \ (g \in G) \,|\, R_g \ (g \in G),\ x_h x'_k = x'_k x_h \ (x, x' \in X,\ (h,k) \in T)\rangle. \qquad (2.11)$$

In particular, C embeds into $A\mathrm{wr}G$.

Let us now assume that G is infinite. Then there exist $g \in G \setminus (G_0 \cup \{1\})$. Clearly we have $(g,1) \notin T$. Consider the free product $A_g * A_1$ of two copies of A defined by the presentations $\langle X_g \,|\, R_g \rangle$ and $\langle X_1 | R_1 \rangle$ respectively. The identity mapping $X_g \cup X_1 \to X_g \cup X_1$ extends to a homomorphism $A_g * A_1 \to C$ since (2.11) contains both R_g and R_1 as relations. Similarly the mapping $x_g \to x_g$, $x_1 \to x_1$, $x_h \to 1$ $(x \in X, h \in G \setminus \{g, 1\})$ extends to an epimorphism $C \to A_g * A_1$. Indeed, under this mapping R_g and R_1 are mapped onto themselves, and all other R_h are mapped onto 1=1. Also, since $(g,1) \notin T$, an arbitrary relation $x_h x'_k = x'_k x_h$ maps onto one of $1 = 1, x_1 = x_1, x_g = x_g$. The composition of the two homomorphisms is the identity mapping on $A_g * A_1$, so that $A_g * A_1$ embeds into C, and hence also into $A\mathrm{wr}G$. However, the relation $x_g x'_1 = x'_1 x_g$ (or equivalently $g^{-1}xgx' = x'g^{-1}xg$) does not hold in $A_g * A_1$, whereas it does hold in $A\mathrm{wr}G$ (actually it is one of the relations in (2.6)). This contradicts the assumption that G is infinite. ∎

We remark that the above proof uses ideas from [3]. Baumslag's Theorem is a special case of Proposition 2.17 but we have avoided the use of free products with amalgamation.

We have now established all the ingredients for the argument given at the beginning of this section, and the proof of Theorem 2.12 is complete.

4 Final Remarks

As remarked in Chapter 1, we may consider the left restricted wreath product, $A\mathrm{wr}_l B$. We have the following analogues to Theorems 2.4 and 2.12:

Theorem 2.18 *Let A and B be monoids, and let G be the group of units of B. Then the wreath product $A\mathrm{wr}_l B$ is finitely generated if and only if both A and B are finitely generated, and either A is trivial, or $B = GV$ for some finite subset V of B.* ∎

Theorem 2.19 *The wreath product $A\mathrm{wr}_l B$ of two monoids A and B is finitely presented if and only if either A is finitely presented and B is finite, or A is trivial and B is finitely presented.* ∎

Note that Theorem 2.19 above is identical to Theorem 2.12, and so $A\mathrm{wr}B$ is finitely presented if and only if $A\mathrm{wr}_l B$ is finitely presented. The analogous statement for finite generation does not hold, however, as shown in the following example:

Example 2.20 Let A be any non-trivial finitely generated monoid, and let B be the monoid defined by the presentation $\langle a, b, c \mid ab = ba = 1, c^2 = ac = c\rangle$. It is easy to see that $B = \{a^i, b^j, ca^i, cb^j : i \geq 0, j \geq 1\}$, and that all these elements are distinct. The group of units of B is $\{a^i, b^j : i \geq 0, j \geq 1\}$, the infinite cyclic group. Clearly $B = \{1, c\}G$, and hence $A\mathrm{wr}B$ is finitely generated. By way of contrast, there is no finite set V such that $B = GV$, and so $A\mathrm{wr}_l B$ is not finitely generated.

Chapter 3

Wreath Products with Finite Top Semigroup

1 Introduction

Over the next two chapters we will attempt to generalize to semigroups the results from Chapter 2. If S and T are arbitrary semigroups, then the absence of an identity particularly in the second component makes the questions of finite generation and finite presentability of their wreath product $S\mathrm{wr}T$ necessarily more complicated than it is for monoids. For this reason we will consider the two cases, where T is finite and where T is infinite, in separate chapters.

In this chapter we look at wreath products $S\mathrm{wr}T$ where T is finite. Since the questions of finite generation and finite presentability of $S\mathrm{wr}T$ are trivial when S and T are both finite, we will generally assume for the remainder of this chapter that S is infinite. Strictly speaking, the wreath product definition in Chapter 1 requires the semigroup S to have an idempotent with respect to which we define the support of a function $T \to S$. However, since T is assumed finite, all functions trivially have finite support, and so we lose nothing by considering the wider class of semigroups $S^T \rtimes T$ where S may or may not contain an idempotent. In an abuse of notation we shall write such a semigroup $S\mathrm{wr}T$. Notice that in this case the semigroups S^T and $S^{(T)}$ are identical.

In Section 3 we will prove the analogue to Theorem 2.4 for semigroups which will require ideas from Chapter 6. Then in Section 4 we prove the analogue to Theorem 2.12 for semigroups which turns out to be very different from the monoid result. First however we give some results on finite generation and presentability of direct products and semi-direct products of semigroups, which will be needed throughout the chapter.

2 Preliminary Results

2.1 Direct Products

We give a couple of results about the finite direct product of copies of a semigroup S, which are immediate consequences of Theorems 1.9 and 1.12 respectively.

Corollary 3.1 *Let S be an infinite semigroup, and let $n \geqslant 2$. Then the direct product of n copies of S, $S^{(n)}$, is finitely generated if and only if S is finitely generated and $S^2 = S$.* ∎

Corollary 3.2 *Let S be an infinite semigroup, and let $n \geqslant 2$. The direct product $S^{(n)}$ is finitely presented if and only if $S^2 = S$, and S is finitely presented and stable.* ∎

2.2 Semi-direct Products

When the semigroup T is finite, the wreath product $S\mathrm{wr}T$ is simply the semi-direct product $S^T \rtimes T$ with the usual action of T on S^T. For this reason we will find it useful to prove some results about general semi-direct products.

Theorem 3.3 *Let A and B be monoids, such that B is finite. Then $A \rtimes B$ is finitely generated if and only if A is finitely generated.*

PROOF. If A is finitely generated, by X say, then clearly the set

$$\{(x, 1_B) : x \in X\} \cup \{(1_A, b) : b \in B\}$$

generates $A \rtimes B$. For the converse, suppose that $A \rtimes B$ is finitely generated by the set $\{(u_i, v_i) : (i \in I)\}$. Let $U = \{u_i : i \in I\}$. We claim that the set $Z = \{\, {}^b u : u \in U, b \in B\}$ generates A. Indeed, given $a \in A$ write $(a, 1_B)$ as a product of generators:

$$(a, 1_B) = (u_1, v_1)(u_2, v_2) \ldots (u_r, v_r).$$

Then

$$a = u_1 \, {}^{v_1}u_2 \ldots {}^{v_1 v_2 \ldots v_{r-1}}u_r \in \langle Z \rangle.$$

Clearly Z is finite, since U and B are finite. Thus A is finitely generated as required. ∎

The following theorem takes presentations for monoids A and B and gives a presentation for the semi-direct product $A \rtimes B$. A consequence is that if both A and B are finitely presented, then so is $A \rtimes B$. For a proof, see [15, Corollary 2].

Theorem 3.4 *Let A and B be monoids defined by the presentations $\langle X \mid P \rangle$ and $\langle Y \mid Q \rangle$ respectively. Let $\phi : B \to \mathrm{End}(A)$ be a homomorphism from B into the* *monoid of endomorphisms of A. For each $x \in X$ and $y \in Y$, ${}^{y}x$ is an element of A and so may be written as a product of generators $w(x, y)$. Then the semi-direct product $A \rtimes_\phi B$ is defined by the presentation*

$$\langle X, Y \mid P, Q, yx = w(x,y)y \ \ (x \in X, y \in Y) \rangle.$$

In particular, if A and B are finitely presented, so is $A \rtimes_\phi B$. ∎

Theorem 3.5 *Let A and B be monoids, where B is finite. Then $A \rtimes B$ is finitely presented only if A is finitely presented.*

Proof. Suppose that the semi-direct product $A \rtimes B$ is finitely presented. Then $A \rtimes B$ is certainly finitely generated, and so by Theorem 3.3, A is finitely generated. So let $\langle X \mid R \rangle$ be a presentation for A with respect to a finite set X, where R may be infinite. Let $\langle B \mid Q \rangle$ be the multiplication table presentation for B. This is a finite presentation, since B is finite. For every $t \in B$ and $x \in X$ choose $\zeta(t, x) \in X^*$ representing the element ${}^{t}x$ in $A \rtimes B$. Then, by Theorem 3.4, we have that $A \rtimes B$ is defined by the (finite) presentation:

$$\langle X, B \mid R,\ Q,\ tx = \zeta(t,x)t \ (a \in X,\ t \in B) \rangle.$$

Notice that the mapping $\xi : X^* \to (X \cup B)^*$ defined by $x \mapsto x$ induces an embedding $A \hookrightarrow A \rtimes B$. Now, since $A \rtimes B$ is finitely presented, there is a finite set $R' \subseteq R$ such that $A \rtimes B$ is defined by the presentation:

$$\langle X, B \mid R',\ Q,\ tx = \zeta(t,x)t \ (x \in X,\ t \in B) \rangle. \tag{3.1}$$

For each $t \in B$ we let $\sigma_t : X^* \to X^*$ be the homomorphism extending the mapping $x \mapsto \zeta(t, x)$, and let $\psi : B^* \to \mathrm{End}(X^*)$ be the homomorphism defined by $t \mapsto \sigma_t$. For $z \in X^*$ and $w \in B^*$ we write $(w\psi)(z)$ as ${}^{w}z$. Notice that, for $z, z_1, z_2 \in X^*$ and $w, w_1, w_2 \in B^*$ we have.

$${}^{w_1 w_2}z \equiv {}^{w_1}({}^{w_2}z), \tag{3.2}$$

since ψ is a homomorphism, and

$$^{w}z_1z_2 \equiv {}^{w}z_1\, {}^{w}z_2, \tag{3.3}$$

since $w\psi$ is an endomorphism. Let A_1 be the monoid defined by the presentation

$$\langle X \mid {}^{t}R',\ {}^{u}x = {}^{v}x\ (x \in X,\ t \in B,\ (u = v) \in Q) \rangle, \tag{3.4}$$

where ${}^{t}R' = \{ {}^{t}u = {}^{t}v : (u = v) \in R' \}$. Since (3.4) is a finite presentation, A_1 is finitely presented.

We now prove that ψ induces a homomorphism $\phi : B \to \mathrm{End}(A_1)$, i.e. that B acts on A_1. We start by proving the following

Claim *If $w \in B^*$, $t \in B$ are such that $w = t$ in B then ${}^{w}x = {}^{t}x$ in A_1 for all $x \in X$. In particular, if $w_1, w_2 \in B^*$ are such that $w_1 = w_2$ in B then ${}^{w_1}x = {}^{w_2}x$ in A_1.*

PROOF. We use induction on the length of w. If $|w| = 1$ then the claim is trivially true. If $|w| = 2$ then ${}^{w}x = {}^{t}x$ is a relation from (3.4) for every $x \in X$. Now suppose that $w \equiv t_1t_2w'$, where $t_1, t_2 \in B$ and $w' \in B^*$. Let $t_3 \in B$ be such that $t_1t_2 = t_3$ in B (this is a relation from Q) and suppose that ${}^{w'}x \equiv x_1x_2 \cdots x_k$ for $x_1, x_2, \ldots, x_k \in X$. Then

$$\begin{aligned}
{}^{w}x &\equiv {}^{t_1t_2w'}x \\
&\equiv {}^{t_1t_2}(x_1x_2\cdots x_k) && \text{(by (3.2))} \\
&\equiv {}^{t_1t_2}x_1\ {}^{t_1t_2}x_2 \cdots {}^{t_1t_2}x_k && \text{(by (3.3))} \\
&= {}^{t_3}x_1\ {}^{t_3}x_2 \cdots {}^{t_3}x_k && ({}^{t_1t_2}x_i = {}^{t_3}x_i \text{ since } (t_1t_2, t_3) \in Q) \\
&\equiv {}^{t_3}(x_1x_2\cdots x_k) && \text{(by (3.3))} \\
&\equiv {}^{t_3}({}^{w'}x) \\
&\equiv {}^{t_3w'}x && \text{(by (3.2))} \\
&= {}^{t}x && \text{(induction hypothesis).}
\end{aligned}$$

This completes the proof of our claim. ∎

We know that B^* acts on X^*, using the homomorphism $\psi : B^* \to \mathrm{End}(X^*)$. We now prove that B^* acts on A_1, that is, for each $w \in B^*$, $w\psi$ induces an endomorphism of A_1. We simply show that given a word $w \in B^*$ and any defining relation $(u = v)$ for A_1 in the presentation (3.4), the corresponding relation ${}^{w}u = {}^{w}v$ also holds in A_1. Take a relation ${}^{t}u = {}^{t}v$ in ${}^{t}R'$, and a word $w \in B^*$. Let $wt = t_1$. Then, using the claim above, we have

$$
\begin{aligned}
{}^{w}({}^{t}u) &= {}^{wt}u = {}^{t_1}u \\
{}^{w}({}^{t}v) &= {}^{wt}v = {}^{t_1}v
\end{aligned}
$$

where ${}^{t_1}u = {}^{t_1}v$ is a relation from ${}^{t_1}R'$. Thus ${}^{w}({}^{t}u) = {}^{w}({}^{t}v)$ is a consequence of the relations in (3.4), and so holds in A_1. Now we take a relation of the form ${}^{u}x = {}^{v}x$, where $(u = v) \in Q$, $x \in X$. Since $u = v$ we must have $wu = wv = t$ say, and then

$$
{}^{wu}x = {}^{t}x = {}^{wv}x,
$$

again using our claim. Thus we have proved that $w\psi$ induces a homomorphism of A_1 for each $w \in B^*$.

Finally, we show that B acts on A_1, that is, for two words $w_1, w_2 \in B^*$ where $w_1 = w_2 \in B$, we have $w_1\psi = w_2\psi$. Equivalently, given $\alpha \in X^*$ we need to show ${}^{w_1}\alpha = {}^{w_2}\alpha$ in A_1. Let $\alpha = x_1 \dots x_k$. Then

$$
{}^{w_1}\alpha \equiv {}^{w_1}x_1 \dots {}^{w_1}x_k = {}^{w_2}x_1 \dots {}^{w_2}x_k \equiv {}^{w_1}\alpha
$$

as required.

We have shown that B acts on A_1, with the action given by ϕ, the homomorphism induced from ψ. Thus we may consider the semi-direct product $A_1 \rtimes_\phi B$ which is defined by the presentation

$$
\langle X, B \mid {}^{t}R',\ Q,\ {}^{u}x = {}^{v}x,\ tx = \zeta(t,x)t\ (x \in X, t \in B, (u = v) \in Q) \rangle. \tag{3.5}
$$

Notice that (3.5) contains (3.1) as a sub-presentation and that all relations from (3.5) hold in $A \rtimes B$. Thus $A \rtimes B \cong A_1 \rtimes_\phi B$ under the homomorphism determined by $x \mapsto x$, $t \mapsto t$. Now, the mapping $\chi : X^* \to (X \cup B)^*$ extending $x \mapsto x$ induces an embedding from A_1 into $A_1 \rtimes_\phi B \cong A \rtimes B$. Thus, the submonoid generated by X in $A_1 \rtimes_\phi B$ is A_1, which is isomorphic to A, the submonoid generated by X in $A \rtimes B$. Therefore A is finitely presented as required. ■

Corollary 3.6 *Let A be an infinite semigroup, and B a finite monoid. Let ϕ be a homomorphism from B to* $\mathrm{End}(A)$. *Then the semi-direct product $A \rtimes_\phi B$ is finitely presented only if A is finitely presented.*

PROOF. Let A^1 denote the monoid formed by adjoining an identity to A. We may extend a given endomorphism $f : A \to A$ to $f' : A^1 \to A^1$ by simply defining $1f = 1$. Then we may extend our homomorphism ϕ to ϕ' in the obvious way: if $b\phi = f$ then define $b\phi' = f'$. Clearly ϕ' is a homomorphism from B to $\mathrm{End}(A^1)$, and the semigroup $A \rtimes_\phi B$ naturally embeds in the monoid $A^1 \rtimes_{\phi'} B$. Now

$$(A^1 \rtimes_{\phi'} B) \setminus (A \rtimes_\phi B) = \{(1, b) : b \in B\}$$

and so $A \rtimes_\phi B$ has finite index in $A^1 \rtimes_{\phi'} B$. Then $A \rtimes_\phi B$ finitely presented implies $A^1 \rtimes_{\phi'} B$ finitely presented, by Theorem 1.7. Then we may apply Theorem 3.5 to see that A^1 is finitely presented. Then again applying Theorem 1.7 we see that A must be finitely presented as required. ■

3 Generators

We have seen that for the wreath product of two semigroups S and T to be finitely generated, it is necessary for both S and T to be finitely generated (see the note after Theorem 2.9. In the case of two groups, this condition is also sufficient. For monoids, an extra condition on T is also required, a condition that is trivially satisfied whenever T is finite. For semigroups, even when T is finite, a little more is required, as the following example shows:

Example 3.7 Let $\mathbb{Z}$ be the group of integers under addition, and let T be the two element right-zero semigroup. Then $\mathbb{Z}$ is finitely generated and T is finite, but $\mathbb{Z}\mathrm{wr}T$ is not finitely generated.

PROOF. Let $Z = \{(f_i, t_i) : i \in I\}$ be a finite subset of $\mathbb{Z}\mathrm{wr}T$. Let $T = \{a, b\}$, so that $ab = b^2 = b$ and $ba = a^2 = a$. Let $F = \mathbb{Z}^T$ be the group of all functions from T into $\mathbb{Z}$. Then, as a set,

$$\mathbb{Z}\mathrm{wr}T = \big\{(f, t) : f \in F, t \in \{a, b\}\big\}.$$

Define a map $\phi : F \to \mathbb{Z}$ by

$$f\phi = af - bf.$$

Clearly ϕ is surjective. Also,

$$(f + g)\phi = a(f + g) - b(f + g) = (af - bf) + (ag - bg) = f\phi + g\phi$$

and hence ϕ is an epimorphism. The inverse image of the identity 0 under ϕ is the set of all constant functions from $\mathbb{Z}$ into T. Define another map $\Phi : \mathbb{Z}\mathrm{wr}T \to \mathbb{Z}$ by

$$(f, t)\Phi = f\phi.$$

Clearly Φ is surjective since ϕ is. That is, $(\mathbb{Z}\mathrm{wr}T)\Phi = \mathbb{Z}$. Now let $g \in F$, $t \in T$. Then

$$x\,{}^t g = (xt)g = tg \;(\forall x \in T),$$

and so ${}^t g$ is constant.

Given two elements (f,t) and (g,u) of $\mathbb{Z}\mathrm{wr}T$ we have $(f,t)(g,u)=(f+{}^{t}g,u)$. Thus

$$\big((f,t)(g,u)\big)\Phi=(f+{}^{t}g,u)\Phi=(f+{}^{t}g)\phi=f\phi+{}^{t}g\phi=f\phi+0=f\phi.$$

Therefore, given any word $w=z_1\ldots z_r$ in Z^+ we have $w\Phi=z_1\Phi$. So $(Z^+)\Phi=Z\Phi$ is necessarily finite (since Z is finite), and so Z^+ cannot be the whole of $\mathbb{Z}\mathrm{wr}T$. Thus $\mathbb{Z}\mathrm{wr}T$ is not finitely generated. ∎

We now state the main theorem of this section. For clarity we divide the theorem into two cases, the cases where the *right diagonal S-act* is finitely generated or not. For the purposes of this chapter, we first give a working definition of this term. For a full explanation and more formal definition see Chapter 6.

Definition 3.8 Let S be a finite semigroup. The *(right) diagonal S-act* of S is the set $S\times S$ with action

$$\big(S\times S\big)\times S\rightarrow\big(S\times S\big)$$

defined by

$$\big((s_1,s_2),s\big)\mapsto\big(s_1s,s_2s\big).$$

The right diagonal S-act, $S\times S$ is generated by a set $X\subseteq S\times S$ whenever $XS^1=S\times S$ using the action above. If such an X can be chosen finite, then S has *finitely generated right S-act.*

Examples of infinite semigroups S with and without finitely generated right S-act are given in Chapter 6. We note here that if S is an infinite semigroup, then the right diagonal S-act $S\times S$ is finitely generated if and only if there is some finite subset U of S satisfying $(U\times U)S=S\times S$. For a proof of this fact, see Theorem 6.14. We will use this equivalent condition for the remainder of this chapter.

Theorem 3.9 *Let S be an infinite semigroup whose right diagonal S-act is* not *finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely generated if and only if*

(i) $S^2 = S$ *and* $T^2 = T$;

(ii) S *is finitely generated;*

(iii) T *has a unique maximal $\mathcal{L}$-class L, the maximal $\mathcal{R}$-classes of T being precisely those $\mathcal{R}$-classes in L.*

Theorem 3.10 *Let S be an infinite semigroup, whose right diagonal S-act is finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely generated if and only if*

(i) $S^2 = S$ *and* $T^2 = T$;

(ii) S *is finitely generated.*

Example 3.11 Let T_1 be a finite left zero semigroup, and T_2 a finite right zero semigroup. Then $S\mathrm{wr}T_1$ is finitely generated for any finitely generated semigroup S, while if S is infinite, $S\mathrm{wr}T_2$ is not finitely generated unless S has finitely generated right diagonal S-act.

Next is a lemma which will be used in the proofs of both the above theorems.

Lemma 3.12 *Let S and T be nontrivial semigroups, at least one of which is infinite. Then $S\mathrm{wr}T$ is finitely generated only if $S^2 = S$ and $T^2 = T$.*

PROOF. Suppose that $S\mathrm{wr}T$ is generated by the finite set $Z = \{(f_i, t_i) : i \in I\}$. Let $s \in S$ and $t \in T$ be arbitrary. Define

$$F = \{f \in S^T : tf = s\}.$$

It is easy to see that F is infinite, and so we may choose $f \in F$ such that $f \neq f_i$ for all $i \in I$. Then (f,t) must be expressible as a product of at least two elements of Z:

$$(f,t) = (f_1,t_1)\dots(f_r,t_r) \quad (r \geqslant 2).$$

Then $f = f_1 \, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$ and so

$$\begin{aligned} s &= tf = tf_1(tt_1)f_2\dots(tt_1\dots t_{r-1})f_r \in S^2, \\ t &= t_1t_2\dots t_r \in T^2, \end{aligned}$$

as required. ∎

For completeness, we restate Theorem 2.9, reformulated for semigroups.

Theorem 3.13 *Let A and B be semigroups. If $A\mathrm{wr}B$ is generated by a set Z then A is generated by the set*

$$X = \{bf : b \in B \text{ and } (\exists c \in B)((f,c) \in Z)\}$$

and B is generated by the set

$$Y = \{b \in B : (\exists f \in A^B)((f,b) \in Z)\}.$$

In particular, if $A\mathrm{wr}B$ is finitely generated then both A and B are also finitely generated. ∎

The following result is crucial in the proof of Theorem 3.9.

Proposition 3.14 *Let S be an infinite finitely generated semigroup satisfying $S^2 = S$ whose right diagonal S-act is not finitely generated. Let T be a finite semigroup. Then the following statements are equivalent.*

(i) *$S\mathrm{wr}T$ is finitely generated.*

(ii) *Given $b, c \in T$ there exists an element $t_{b,c} \in T$ satisfying:*

(a) $xt_{b,c} \neq bt_{b,c}$ *if* $x \in T \setminus \{b\}$

(b) $c \in t_{b,c}T$

(iii) *Given $c \in T$ there exists $t \in T$ satisfying:*

(a) *t acts injectively by right multiplication.*

(b) $c \in tT$.

(iv) *Given $c \in T$ there exists a right identity e for T such that $c \in eT$.*

PROOF. When $T = \{1\}$ all the conditions are trivially satisfied, and so we assume $T \neq \{1\}$. To show (i) implies (ii) we use the assumption that $S \times S$ is not a finitely generated right S-act. Suppose $S\mathrm{wr}T$ is finitely generated by $\{(f_i, t_i) : i \in I\}$. Then we set

$$U = \bigcup_{i \in I} (T) f_i.$$

Since each f_i has finite image, U must be finite, and so $U \times U$ cannot generate $S \times S$ as a right S-act. We choose $(s_1, s_2) \in (S \times S) \setminus ((U \times U)S)$. Now given b, c as in (ii) we define a function $f : T \to S$ as follows:

$$xf = \begin{cases} s_1 & \text{if } x = b \\ s_2 & \text{if } x \neq b. \end{cases}$$

By hypothesis, we may assume $(f, c) = (f_1, t_1) \dots (f_r, t_r)$. We claim that t_1 satisfies the conditions for $t_{b,c}$. Clearly $c \in t_1 T$ since $c = t_1 \dots t_r$, and so (ii)(b) holds. Suppose that $bt_1 = xt_1$ for some $x \neq b$. Using $f = f_1 \, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$ we see that

$$s_1 = (b)f = (b)f_1(bt_1)f_2 \dots (bt_1 \dots t_{r-1})f_r,$$

$$s_2 = (x)f = (x)f_1(xt_1)f_2 \dots (xt_1 \dots t_{r-1})f_r,$$

so that $(s_1, s_2) = (u_1, u_2)k$ where $u_1 = (b)f_1$, $u_2 = (x)f_1$ and

$$k = (bt_1)f_2 \dots (bt_1 \dots t_{r-1})f_r = (xt_1)f_2 \dots (xt_1 \dots t_{r-1})f_r$$

which contradicts our choice of s_1 and s_2. Thus the assumption that $bt_1 = xt_1$ was false, and t_1 does satisfy (ii)(a) as claimed.

Given that (ii) is true, let $c \in T$. Choose $b \in T$ arbitrarily. We let P be the subset of T consisting of all elements t satisfying $c \in tT$ and $bt \neq xt$ for $x \neq b$. Condition (ii) shows that P is non-empty. We just need to show that P contains an element t which acts injectively by right multiplication. Note that t acting injectively is equivalent to $|Tt| = |T|$. Let p be an element of P such that $|Tp|$ is maximal. Then $2 \leqslant |Tp| \leqslant |T|$. For a contradiction, assume that $|Tp| < |T|$. Let Q be a subset of T such that $|Q| = |Tp|$ and $Qp = Tp$. We choose $d \in T \setminus Q$ and apply (ii) with $b = d$ and $c = p$. Then we may find t' such that

1. $dt' \neq xt'$ for $x \neq d$

2. $p \in t'T$.

Set $Q' = Q \cup \{d\}$. Then $|Tt'| \geqslant |Q't'| = |Q| + 1 = |Tp| + 1$ since $dt' \neq xt'$ for $x \in Q$ and if $b_1, b_2 \in Q$ then $b_1t' = b_2t'$ implies $b_1p = b_2p$ and so $b_1 = b_2$. This gives us a contradiction. Thus there must be some element of P that acts injectively by right multiplication, and we have proved (iii).

To show (iii) implies (iv) we simply note that since T is finite, there exists a natural number n such that t^n is an idempotent. Then t is injective implies $Tt = T$ which implies that $Tt^n = T$, and t^n is a right identity for T. Also $c \in tT$ implies $c = tz$ for some z, and so $c = t^n tz \in t^n T$ as required, and we set $e = t^n$.

We now show that (iv) implies (i). Since S is finitely generated, and $S^2 = S$ we have by Corollary 3.1 that $S \times \dots \times S$ ($|T|$ copies) is also finitely generated. Let X be some such generating set. Then it is easy to see that the set

$$\{(f, t) : f \in X, t \in T\}$$

generates $S\mathrm{wr}T$. Given an element (g, t) of $S\mathrm{wr}T$ we consider g as an element of $S \times \ldots \times S$ and write it in terms of the generators X, $g = f_1 \ldots f_n$ say. We use (iv) to find a right identity $e \in T$ such that $t \in eT$. Then $(g, t) = (f_1, e) \ldots (f_{n-1}, e)(f_n, q)$ where q is chosen such that $eq = t$. ■

Condition (iv) in the last proposition tells us quite a lot about the structure of T if $S\mathrm{wr}T$ is to be finitely generated. We make some simple observations about finite semigroups satisfying this condition. Some will be needed in the proof of Theorem 3.9, others will be useful in the next section.

Lemma 3.15 *Suppose d and e are right identities for a semigroup T, and that $d \leqslant_{\mathcal{R}} e$. Then $d = e$.*

PROOF. Suppose $ex = d$. Then $dx = dex$ since e is a right identity, and $dex = d^2 = d$. So $e = ed = edx = ex = d$ as required. ■

Proposition 3.16 *Let T be a finite semigroup satisfying condition (iv) in Proposition 3.14, namely that each element of T lies in the principal right ideal generated by some right identity. Then we know the following facts about T:*

(i) *T has a unique maximum $\mathcal{L}$-class L, which contains every right identity.*

(ii) *There is a unique right identity e_i in each maximal $\mathcal{R}$-class R_i, and these are the only right identities of T.*

(iii) *Given maximal $\mathcal{R}$-classes R_i and R_j we have*

$$R_i R_j \subseteq R_i.$$

(iv) *Each maximal $\mathcal{R}$-class R_i is in fact a group, with identity e_i.*

PROOF. (i) Let e be some right identity for T. Given $z \in T$ we have $e \leqslant_{\mathcal{L}} ze = z$. Thus there must be a unique maximal $\mathcal{L}$-class, containing e.

(ii) Take $x \in R_i$. By hypothesis $x \in e_iT$ for some right identity e_i, and then $e_i \geqslant_{\mathcal{R}} x$ implies $e_i \in R_i$. Then using Lemma 3.15 and the fact that T is finite, we see that apart from one for each maximal $\mathcal{R}$-class, no other right identities can exist.

(iii) Take $x \in R_i$, $y \in R_j$. By (ii) we know there exists a right identity $e_j \in R_j$.Thus $yz = e_j$ for some $z \in T$. Then

$$(xy)z = x(yz) = xe_j = x$$

and so $xy \geqslant_{\mathcal{R}} x$ which implies $xy\mathcal{R}x$. Thus $xy \in R_i$ as required.

(iv) Each R_i contains a right identity e_i, and every element in R_i is right invertible (to e_i). Then R_i is a group by Theorem 1.1. ∎

Proposition 3.17 *Let T be a finite semigroup satisfying $T^2 = T$. Then the following conditions are equivalent:*

(i) *Each element of T lies in the principal right ideal generated by some right identity of T.*

(ii) *T has a unique maximal $\mathcal{L}$-class L, and the maximal $\mathcal{R}$-classes R_i of T are precisely those $\mathcal{R}$-classes in L.*

PROOF. Suppose that (i) holds. By the last proposition, T has a unique maximal $\mathcal{L}$-class, containing the right identities of T. Each maximal $\mathcal{R}$-class R_i is a group containing a right identity e_i. The elements of the group R_i must be $\mathcal{L}$-related to e_i, and so lie in L. It remains to show that each element of L is contained in some maximal $\mathcal{R}$-class R_i. Given $y \in L$ we must have $y\mathcal{L}e_i$ and so $zy = e_i$ for some $z \in T$. Since e_i acts injectively by right multiplication on T, so must

y. Some power y^m of y must therefore be a right identity, using the argument from the proof of Proposition 3.14. Now $y\mathcal{R}y^m$, and y^m, as a right identity, lies in some R_i. Thus $y \in R_i$ as required.

To prove the converse, suppose that (ii) holds. Given $x \in T$, we may choose y in some maximal $\mathcal{R}$-class R_i such that $y \geqslant_{\mathcal{R}} x$. By hypothesis, each R_i is contained in L, so $y \in L$. Since L lies above every $\mathcal{L}$-class of T, we must have $T \setminus \{y\} \subseteq Ty$. We claim that in fact $Ty = T$. Since $T^2 = T$ we can write $y = ab$ for $a, b \in T$. If $b = y$ then $y = ay \in Ty$. Otherwise $b \in Ty$ and we may write $b = qy$ for some $q \in T$. Then $y = ab = (aq)y \in Ty$. In either case $y \in Ty$ and our claim is proved. This means that y acts injectively by right multiplication. As observed above, some power y^m of y is a right identity for T, and lies in the same $\mathcal{R}$ and $\mathcal{L}$-class as y. Thus $e = y^m \geqslant_{\mathcal{R}} x$ as required. ∎

We are now in a position to prove Theorem 3.9, restated below:

Theorem 3.9 *Let S be an infinite semigroup whose right diagonal S-act is* not *finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely generated if and only if*

(i) $S^2 = S$ *and* $T^2 = T$*;*

(ii) S *is finitely generated;*

(iii) T *has a unique maximal $\mathcal{L}$-class L, the maximal $\mathcal{R}$-classes of T being precisely those $\mathcal{R}$-classes in L.*

Proof. If $S\mathrm{wr}T$ is finitely generated, then Lemma 3.12 and Theorem 3.13 tell us that S is finitely generated, and that $S^2 = S$ and $T^2 = T$. The conditions in Proposition 3.14 are then satisfied, and so we have by condition (iv) that every element of T lies in the right ideal generated by some right identity. Then applying Proposition 3.17 gives us the result.

On the other hand, if conditions (i)–(iii) hold then we use Proposition 3.17 to show that every element of T lies in the right ideal generated by some right identity. Then we use Proposition 3.14 to conclude that $S\mathrm{wr}T$ is finitely generated. ■

We finish the section by proving Theorem 3.10, restated below:

Theorem 3.10 *Let S be an infinite semigroup, whose right diagonal S-act is finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely generated if and only if*

(i) *$S^2 = S$ and $T^2 = T$;*

(ii) *S is finitely generated.*

PROOF. The direct part of the proof follows simply from Lemma 3.12 and Theorem 3.13. For the converse, assume that conditions (i) and (ii) hold. Suppose $|T| = n$. By Proposition 6.15 we know that the assumption that $S \times S$ is finitely generated as a right S-act implies that $S^{(n)}$ is also finitely generated as a right S-act. We may assume that $S^{(n)} = U^{(n)}S$ for some finite subset U of S. Let F be the set of all functions from T into U. Suppose S is generated by a finite subset X, and let H be the set of all constant functions from T into X. Since T, U and X are all finite, so are F and H. We claim that $S\mathrm{wr}T$ is finitely generated by the set

$$Z = \big\{(f, t) : f \in F \cup H, t \in T\big\}.$$

To generate an element (f, t) of $S\mathrm{wr}T$ we think of f as an element of $S^{(n)}$. That $S^{(n)} = U^{(n)}S$ means precisely that we may write $f = gs$ for some $g \in F$, and $s \in S$. Suppose $s = x_1x_2 \dots x_m$ for $x_i \in X$. We use the fact that $T^2 = T$ to write t as a product of $m + 1$ elements of T. Let $t = t_1t_2 \dots t_{m+1}$ and let h_i denote the constant map from T into x_i. Clearly ${}^qh_i = h_i$ for all $q \in T$. Then

$$(f, t) = (g, t_1)(h_1, t_2) \dots (h_m, t_{m+1}) \in \langle Z\rangle$$

so $S\mathrm{wr}T$ is finitely generated as required. ■

4 Finite Presentability

This section is dedicated to proving the following:

Theorem 3.18 *Let S be a semigroup such that S does not have finitely generated diagonal S-act, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely presented if and only if*

(i) *S is stable;*

(ii) *$S^2 = S$;*

(iii) *$T = M \times E$ is the direct product of a monoid and a left zero semigroup.*

Example 3.19 If T is a finite monoid then T satisfies (iii) where E is the trivial left zero semigroup. Therefore $S\mathrm{wr}T$ is finitely presented for any stable semigroup S. If S is a finitely presented monoid, then conditions (i) and (ii) trivially hold, and the theorem reduces to Theorem 2.12 for monoids.

Example 3.20 Let $S = \mathbb{N}$ be the monoid of natural numbers. Let T be the semigroup defined by the presentation

$$\langle a, b, z \mid a^2 = ab = a, b^2 = ba = b, za = az = z^2 = bz = zb = z \rangle.$$

T is simply the two element left zero semigroup, with a multiplicative zero adjoined. Then T has a unique maximal $\mathcal{L}$-class $L = \{a, b\}$, and the maximal $\mathcal{R}$-classes of T ($\{a\}$ and $\{b\}$) are precisely those contained in L. Obviously $T^2 = T$, and so S and T satisfy the conditions in Theorem 3.9. But T is not the direct product of a monoid and a left zero semigroup, and so $S\mathrm{wr}T$ is finitely generated, but is not finitely presented.

In order to prove Theorem 3.18 we will first prove Theorem 3.21 below, and then show in Proposition 3.27 that the necessary and sufficient conditions in Theorems 3.18 and 3.21 are in fact equivalent.

Theorem 3.21 *Let S be a semigroup such that S does not have finitely generated diagonal S-act, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely presented if and only if*

(i) *S is stable;*

(ii) *$S^2 = S$ and $T^2 = T$;*

(iii) *T has a unique maximal $\mathcal{L}$-class L, the maximal $\mathcal{R}$-classes of T being precisely those $\mathcal{R}$-classes in L;*

(iv) *Every $\mathcal{R}$-class of T lies below one and only one maximal $\mathcal{R}$-class. So if a and b lie in maximal $\mathcal{R}$-classes, and $c \leqslant_{\mathcal{R}} a$, $c \leqslant_{\mathcal{R}} b$ then we must have $a\mathcal{R}b$.*

The proof of Theorem 3.21 is quite technical, and so we will prove the direct and converse parts separately, in Theorems 3.25 and 3.26 respectively. We start with some simple results that will be used in the proofs.

Lemma 3.22 *Suppose T is a finite semigroup and x is a (right) injective element such that $x \leqslant_{\mathcal{R}} e$ for some right identity $e \in T$. Then in fact $x\mathcal{R}e$.*

Proof. Since x is injective, we also have that x^n is injective for all $n \in \mathbb{N}$. But since T is finite, we may find n such that x^n is idempotent. Any idempotent injective element in a finite semigroup must be a right identity, so x^n is a right identity. Clearly $x\mathcal{R}x^n$, so $x^n \leqslant_{\mathcal{R}} e$. Lemma 3.15 gives us that $x^n = e$, and so we must have that $x\mathcal{R}e$ as required. ∎

We note that Lemma 3.22 fails when T is infinite. Indeed, in the monoid $\mathbb{N}$ of natural numbers, every element x is right injective and satisfies $x \leqslant_{\mathcal{R}} 1_{\mathbb{N}}$, but the identity of $\mathbb{N}$ is $\mathcal{R}$-related only to itself.

Lemma 3.23 *$\mathcal{R}$-classes preserve (right) injectivity.*

PROOF. If $a\mathcal{R}b$ then clearly $pa = qa$ if and only if $pb = qb$. ∎

Lemma 3.24 *Let T be a finite semigroup satisfying $T^2 = T$ and condition (iii) of Theorem 3.21. Then*

(i) *The maximal $\mathcal{L}$-class L is a subsemigroup of T, and in fact $L = R_1 \cup \ldots \cup R_n$ is a band of groups.*

(ii) *$T \setminus L$ is an ideal of T.*

PROOF. (i) As observed in Proposition 3.16, the R_i are groups satisfying

$$R_i R_j \subseteq R_i.$$

Since $L = R_1 \cup \ldots \cup R_n$ the result follows.

(ii) $T \setminus L$ is the subset of all elements of T that are neither $\mathcal{L}$-maximal nor $\mathcal{R}$-maximal. Given $z \in T \setminus L$ and $x \in T$ we have $z \geqslant_{\mathcal{L}} xz$ and $z \geqslant_{\mathcal{R}} zx$, so neither xz nor zx are $\mathcal{L}$ or $\mathcal{R}$-maximal. Thus xz and zx lie in $T \setminus L$ as required. ∎

We may now prove the direct implication of Theorem 3.21.

Theorem 3.25 *Let S be a semigroup such that S does not have finitely generated diagonal S-act, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}T$ is finitely presented implies*

(i) *S is stable;*

(ii) *$S^2 = S$ and $T^2 = T$;*

(iii) *T has a unique maximal $\mathcal{L}$-class L, the maximal $\mathcal{R}$-classes of T being precisely those in L;*

(iv) *Every $\mathcal{R}$-class of T lies below one and only one maximal $\mathcal{R}$-class. So if a and b lie in maximal $\mathcal{R}$-classes, and $c \leqslant_{\mathcal{R}} a$, $c \leqslant_{\mathcal{R}} b$ then we must have $a\mathcal{R}b$.*

PROOF. That (ii) and (iii) hold follows from Theorem 3.9, since $S\mathrm{wr}T$ is finitely generated. Now $S\mathrm{wr}T$ is simply the semi-direct product $S^T \rtimes T$, and so this must be finitely presented. Let X be the monoid obtained by adjoining an identity to S^T. We may extend the action of T on S^T to X by defining ${}^t1 = 1$, and form the semidirect product $X \rtimes T$. Now $S^T \rtimes T$ is a subsemigroup of finite index in $X \rtimes T$, since T is finite. Thus $X \rtimes T$ is finitely presented if and only if $S^T \rtimes T$ is finitely presented.

Since we have already shown (ii) and (iii), we know that T has a unique maximal $\mathcal{L}$-class L, and Lemma 3.24 tells us that the complement of L in T is an ideal of T. Thus the complement of $X \rtimes L$ is an ideal of $X \rtimes T$, and so using Theorem 1.6 $X \rtimes L$ must also be finitely presented. Again by Lemma 3.24, we know that $L = R_1 \cup \ldots \cup R_n$ is a band of groups, and so

$$X \rtimes L = (X \rtimes R_1) \cup \ldots \cup (X \rtimes R_n)$$

is a band of monoids. Then [1, Corollary 5.6] tells us that $X \rtimes L$ is finitely presented only if each $X \rtimes R_i$ is finitely presented. Now $X \rtimes R_i$ is the semidirect product of an infinite semigroup by a group, and so we may apply Corollary 3.6 to see that X must also be finitely presented. Then S^T which is a subsemigroup of X of finite index must also be finitely presented, and so S must be stable. Thus (i) holds.

Suppose that not every $\mathcal{R}$-class lies underneath a unique maximal $\mathcal{R}$-class. Since T is finite, that means that there exists an $\mathcal{R}$-class, $\mathcal{R}_y$ say, that lies underneath two distinct maximal $\mathcal{R}$-classes. Let K be the set of all elements of T that do not lie underneath a unique maximal $\mathcal{R}$-class. Choose z to be maximal in K under the $\mathcal{R}$-ordering. (Clearly K is finite so we can do this). Obviously z does not lie in a maximal $\mathcal{R}$-class. Proposition 3.16 tells us that there is a right identity e_z lying in a maximal $\mathcal{R}$-class above z. If z were injective, Lemma 3.22 would give us that $z\mathcal{R}e_z$, which is a contradiction. Thus z is not injective. Using (iii) we may let d and e be right identities from two of the (possibly several) distinct maximal $\mathcal{R}$-classes that are above z.

Since $S\mathrm{wr}T$ is finitely presented, by Theorem 3.9 S is finitely generated, and $S^2 = S$. Thus $S \times \ldots \times S$ is finitely generated, so we may choose for it a finite generating set F say. From the proof of Proposition 3.14 we know that $S\mathrm{wr}T$ is generated by the set $\{(f,t) : f \in F, t \in T\}$, and so can be finitely presented by

$$\langle \{(f,t) : f \in F, t \in T\} \mid R \rangle,$$

for some finite set of relations R. Let $N \in \mathbb{N}$ be larger than the length of any word in any relation in R. Let $X = \bigcup_{f\in F}(T)f$. Then X is finite and $S = \langle X \rangle$. Let $U \subseteq S$ be the finite set of all words of length $\leqslant N+3$ in X^+. Since $S \times S$ is not a finitely generated S-act, we may choose $(s_1, s_2) \in (S \times S) \setminus ((U \times U)S^1)$. Now $z <_{\mathcal{R}} d$ and $z <_{\mathcal{R}} e$ so we may choose $p, q \in T$ such that $z = dp = eq$. Also, z is not injective, so we may choose $a \neq b \in T$ such that $az = bz$.

Let $g : T \to S$ be defined by

$$(x)g = \begin{cases} s_1 & \text{if } x = a \\ s_2 & \text{if } x \neq a. \end{cases}$$

We can write $g = f_1 \ldots f_r$ $(f_i \in F)$ since F generates $S \times \ldots \times S$, and we note that by choice of s_1, s_2, g cannot be expressed as a word in F of less than $N+4$

generators. Then

$$(f_1, d)\dots(f_{r-1}, d)(f_r, p) = (g, z) = (f_1, e)\dots(f_{r-1}, e)(f_r, q)$$

in $S\mathrm{wr}T$. We just need to show that this cannot be deduced from relations in R (to get our contradiction).

Suppose $(f_1, d)\dots(f_r, p) \equiv \gamma_1, \gamma_2, \dots, \gamma_n \equiv (f_1, e)\dots(f_r, q)$ is an elementary sequence for this relation. Then $|\gamma_i| \geqslant N + 4$, since g cannot be expressed as a word in F of smaller size. Let α_i be the word of equivalent length to γ_i obtained by taking just the second (T) components. So $\alpha_1 = d^{r-1}p$, $\alpha_n = e^{r-1}q$. Also, write $\overline{\alpha_i}$ to be the prefix of α_i of length N, and $\underline{\alpha_i}$ to be the prefix of α_i of length 1. So $\underline{\alpha_1} = d$, $\underline{\alpha_n} = e$.

We claim that $z <_{\mathcal{R}} \overline{\alpha_i} \leqslant_{\mathcal{R}} d$ for $1 \leqslant i \leqslant n$, which would mean that $\overline{\alpha_n} \not\leqslant_{\mathcal{R}} e$. ($\overline{\alpha_n}$ cannot lie below two maximal $\mathcal{R}$-classes, since $\overline{\alpha_n} >_{\mathcal{R}} z$). Thus $\underline{\alpha_n} \neq e$, which is our contradiction. So it just remains to prove this claim.

Since $\alpha_i = z$ in T, we have $\overline{\alpha_i} \geqslant_{\mathcal{R}} z$ for all i. Suppose we have $\overline{\alpha_i}\mathcal{R}z$ for some i. Let $\alpha_i = t_1 \dots t_N \dots t_{N+M}$ where we have seen that $M \geqslant 4$. Suppose

$$\gamma_i = (h_1, t_1)\dots(h_N, t_N)\dots(h_{N+M}, t_{N+M}).$$

Then

$$g = h_1\, {}^{t_1}h_2 \dots {}^{t_1 \dots t_{N-1}}h_N\, {}^{t_1 \dots t_N}h_{N+1} \dots {}^{t_1 \dots t_{N+M-1}}h_{N+M},$$

and $z\,\mathcal{R}\,\overline{\alpha_i} = t_1 \dots t_N$, so

$$(t_1 \dots t_{N+i})\mathcal{R}z \quad (0 \leqslant i \leqslant M).$$

Let

$$\phi = {}^{t_1 \dots t_N}h_{N+1} \dots {}^{t_1 \dots t_{N+M-1}}h_{N+M}.$$

Then $a\phi = b\phi$ since all the superscripts lie in the same $\mathcal{R}$-class as z - use Lemma 3.23. Then

$$\begin{aligned} s_1 &= (a)g = (a)h_1 \dots (a)\, {}^{t_1 \dots t_{N-1}}h_N s = \quad u_1 s \\ s_2 &= (b)g = (b)h_1 \dots (b)\, {}^{t_1 \dots t_{N-1}}h_N s = \quad u_2 s \end{aligned}$$

where $s = a\phi = b\phi$ and u_1, u_2 both have length $\leqslant N$. This is a contradiction, since we chose $(s_1, s_2) \notin (U \times U)S^1$. So $\overline{\alpha_i} >_{\mathcal{R}} z$ for all $1 \leqslant i \leqslant n$. It just remains to show that $\overline{\alpha_i} \leqslant_{\mathcal{R}} d$ for all i.

Now $\overline{\alpha_1} \leqslant_{\mathcal{R}} \underline{\alpha_1}$ and $\underline{\alpha_1}\, \mathcal{R}\, d$ so this is true for $i = 1$. For a contradiction choose $j \in \{1, \ldots, n\}$ minimal such that $\overline{\alpha_j} \not\leqslant_{\mathcal{R}} d$. Thus $\underline{\alpha_j} \not\leqslant_{\mathcal{R}} d$, since $\overline{\alpha_j} \leqslant_{\mathcal{R}} \underline{\alpha_j}$.

Now $z <_{\mathcal{R}} \overline{\alpha_{j-1}} \leqslant_{\mathcal{R}} d$ by choice of j. Let $z <_{\mathcal{R}} \underline{\alpha_{j-1}} \leqslant_{\mathcal{R}} e'$ for e' in some maximal $\mathcal{R}$-class. Then $\overline{\alpha_{j-1}} \leqslant_{\mathcal{R}} \underline{\alpha_{j-1}} \leqslant_{\mathcal{R}} e'$, and since $\overline{\alpha_{j-1}}$ lies under e' and d, and is strictly greater than z, we must have $d = e'$. So $z <_{\mathcal{R}} \underline{\alpha_{j-1}} \leqslant_{\mathcal{R}} d$.

So $\underline{\alpha_{j-1}} \neq \underline{\alpha_j}$, and thus when $\gamma_{j-1} \to \gamma_j$, the first generator must change. Let $\gamma_{j-1} \equiv v'\beta'$, $\gamma_j \equiv w'\beta'$ for $(v', w') \in R$. Taking T components we get $\alpha_{j-1} = v\beta$, $\alpha_j = w\beta$, and $|v|, |w| < N$ by choice of N. Thus $\overline{\alpha_{j-1}} = v\beta_1$, $\overline{\alpha_j} = w\beta_2$ for prefixes β_1 and β_2 of β.

Case 1: $\beta_1 \leqslant_{\mathcal{R}} \beta_2$.
Then $w\beta_2 \geqslant_{\mathcal{R}} v\beta_1$. We must have $e' \geqslant_{\mathcal{R}} \overline{\alpha_j} = w\beta_2 \geqslant_{\mathcal{R}} v\beta_1 = \overline{\alpha_{j-1}} >_{\mathcal{R}} z$ for e' in some maximal $\mathcal{R}$-class. Then again $\overline{\alpha_{j-1}}$ lies below both d and e', and is strictly greater than z, so we must have that $d = e'$. This gives us our contradiction - $\overline{\alpha_j} \leqslant_{\mathcal{R}} d$.

Case 2: $\beta_1 >_{\mathcal{R}} \beta_2$.
Then $v\beta_1 >_{\mathcal{R}} w\beta_2$, and so $\overline{\alpha_j} = w\beta_2 <_{\mathcal{R}} v\beta_1 = \overline{\alpha_{j-1}} \leqslant_{\mathcal{R}} d$, which is a contradiction.

In either case we get a contradiction, and so our original assumption was wrong. Thus $\overline{\alpha_i} \leqslant_{\mathcal{R}} d$ for all i as required, and condition (iv) holds. ■

We now prove the reverse implication part of Theorem 3.21.

Theorem 3.26 *Let S be an infinite and T a finite semigroup. Then $S\mathrm{wr}T$ is finitely presented whenever*

(i) *S is stable;*

(ii) $S^2 = S$ *and* $T^2 = T$*;*

(iii) *For every* $c \in T$ *there exists a right identity* $e \in T$ *such that* $c \in eT$*;*

(iv) *Every* $\mathcal{R}$*-class of* T *lies below one and only one maximal* $\mathcal{R}$*-class. So if* a *and* b *lie in maximal* $\mathcal{R}$*-classes, and* $c \leqslant_{\mathcal{R}} a$*,* $c \leqslant_{\mathcal{R}} b$ *then we must have* $a\mathcal{R}b$*.*

PROOF. Suppose conditions (i)-(iv) hold. That S is finitely generated, together with (ii) and (iii) gives us that $S\mathrm{wr}T$ is finitely generated (using Proposition 3.17 and Theorem 3.9). We may assume that $Z = \{(f,t) : f \in X, t \in T\}$ is a finite generating set, for some finite set $X \subseteq S \times \ldots \times S$. We may further assume that ${}^tX \subseteq X$ for all $t \in T$, since T is finite.

Clearly we must have that X generates $S \times \ldots \times S$, since any $f \in S \times \ldots \times S$ can be written $f = f_1 \, {}^{t_1}f_2 \ldots {}^{t_1 \ldots t_{r-1}}f_r$ where f_1 and ${}^tf_i$ are elements of X by assumption.

Since $S^2 = S$ and S is stable, we must have that $S \times \ldots \times S$ is finitely presented - by $\langle X \mid R_X \rangle$ say. We now construct a finite presentation for $S\mathrm{wr}T$ and prove that it is indeed a presentation by giving an attainable normal form for the elements.

By condition (iv) and Proposition 3.16, each element $t \in T$ lies below exactly one right identity. Denote this by e_t, and choose an element $e_t^{-1}t \in T$ such that $e_t(e_t^{-1}t) = t$.

We will show how to define finite sets of relations $R_1, \ldots, R_4$ on the set of generators Z such that $\langle Z \mid R_1, R_2, R_3, R_4 \rangle$ defines $S\mathrm{wr}T$.

We start by defining R_1. Given two elements $(f,t),(g,s) \in Z$ we define a relation

$$(f,t)(g,s) = (f,e_t)({}^tg,(e_t^{-1}t)s)$$

in R_1. This gives us at most $|X|^2|T|^2$ relations, so R_1 is finite. Clearly these relations hold in $S\mathrm{wr}T$.

Given two elements of the form (f,d), $(g,e) \in Z$ where d and e are right identities for T, we define a relation $(f,d)(g,e) = (f,d)(g,d)$ in R_2. Clearly these relations also hold in $S\mathrm{wr}T$, and R_2 is finite.

Now for R_3 we simply modify the relations R_X. Each relation of R_X gives rise to a finite number of relations in R_3, and so R_3 is also finite. Suppose $u_1u_2\dots u_m = v_1v_2\dots v_n$ is a relation in R_X.

- If $m = n = 1$ then we let $(u_1,t) = (v_1,t)$ be relations in R_3 for each $t \in T$.
- If $m \geqslant 2, n = 1$ then we let $(u_1,e_i)(u_2,e_i)\dots(u_{m-1},e_i)(u_m,t) = (v_1,e_it)$ be relations in R_3 for each right identity e_i and each $t \in T$.
- If $m,n \geqslant 2$ then we let

  $$(u_1,e_i)(u_2,e_i)\dots(u_{m-1},e_i)(u_m,t) = (v_1,e_i)(v_2,e_i)\dots(v_{n-1},e_i)(v_n,t)$$

  be relations in R_3 for each right identity e_i and each $t \in T$.

It is easy to see that the relations from R_3 all hold in $S\mathrm{wr}T$.

Now we may have that $et_1 = et_2$ for some right identity e and some $t_1 \neq t_2 \in T$. For each such occurrence define a relation

$$(f,e)(g,t_1) = (f,e)(g,t_2)$$

in R_4 for every $f,g \in X$. Then R_4 is finite, and the relations clearly hold in $S\mathrm{wr}T$.

Next we show we can modify any word in the generators Z to give a normal form for $S\mathrm{wr}T$. Given a word $(x_1,t_1)(x_2,t_2)\dots(x_n,t_n)$ we first apply $n-1$ relations from R_1 to obtain an equivalent word $(x_1,e_1)(x_2',e_2)\dots(x_{n-1}',e_{n-1})(x_n',t')$.

The right identity e_1 will clearly be uniquely defined by the T component of our word, in fact $e_1 = e_{t_1}$. Then we may apply relations from R_2 and set each $e_j = e_1$ for $2 \leqslant j \leqslant n-1$.

Now that all but the last of the T-component entries are right identities for T, we see that the X components multiply normally (as though in a direct product). For example $(f, e)(g, t) = (f\,{}^{e}g, et) = (fg, et)$. So now modified relations from R_X that make up R_3 may be used to put the word in X in a normal form.

Then finally we eliminate the possible ambiguity where $e_1 t = e_1 t'$ with relations from R_4.

Our normal form for any word in $S\mathrm{wr}T$ equal to (f, t) consists of a normal form for f in the generators X in the left hand component, with all right hand components but the last equal to a specific fixed right identity (depending on t), and a unique 'correcting factor' in the last right hand component. ■

Theorems 3.25 and 3.26 together prove Theorem 3.21. To prove the main Theorem of this section it remains to prove the following proposition, which shows that the conditions in Theorems 3.18 and 3.21 are equivalent:

Proposition 3.27 *Let T be a finite semigroup. Then the following conditions are equivalent:*

(i) *$T = M \times E$ is the direct product of a monoid and a left zero semigroup.*

(ii) *T satisfies the following:*

(a) *$T^2 = T$;*

(b) *T has a unique maximal $\mathcal{L}$-class L, the maximal $\mathcal{R}$-classes of T being precisely those $\mathcal{R}$-classes in L;*

(c) *Every $\mathcal{R}$-class of T lies below one and only one maximal $\mathcal{R}$-class.*

PROOF. Suppose (ii) holds. By Proposition 3.17, using (a) and (b), we know that every element of T lies in the principal right ideal generated by some right identity. Let $e_1, \ldots, e_n$ be the right identities for T. Let $M_i = e_iT$ for $1 \leqslant i \leqslant n$. Now $(e_ip)(e_iq) = e_ipq \in e_iT$ and so M_i is a subsemigroup of T, in fact M_i is a monoid with identity e_i. Suppose that $x \in M_i \wedge M_j$. Then $x \leqslant_{\mathcal{R}} e_i$ and $x \leqslant_{\mathcal{R}} e_j$. By Proposition 3.16(ii) we know that each e_i lies in a maximal $\mathcal{R}$-class, so using (c) we must have that $e_i\mathcal{R}e_j$. Again by Proposition 3.16(ii) we must have $e_i = e_j$. Since every element of T must lie in some M_i, T must be the disjoint union of the monoids M_i. Let $E = \{e_1, \ldots, e_n\}$. Then E is a left zero semigroup of order n. Define a map $\phi : T \to M_1 \times E$ as follows:

$$(e_it)\phi = (e_1t, e_i).$$

This is well defined, since $e_it = e_js$ implies

$$e_1t = e_1e_it = e_1e_js = e_1s,$$

and also that $i = j$ since M_i and M_j are disjoint. Conversely, if $(e_it)\phi = (e_js)\phi$ then $i = j$ and $e_1t = e_1s$. Thus

$$e_it = e_ie_1t = e_ie_1s = e_is = e_js,$$

and ϕ is injective. Now

$$(e_it_1e_jt_2)\phi = (e_it_1t_2)\phi = (e_1t_1t_2, e_i) = (e_1t_1, e_i)(e_1t_2, e_j) = (e_it_1)\phi(e_jt_2)\phi$$

and so ϕ is a homomorphism. Clearly ϕ is surjective, and so ϕ is an isomorphism as required.

Now suppose that (i) holds, that $T = M \times E$. For any monoid M and any left zero semigroup E, we must have $M^2 = M$ and $E^2 = E$. Hence $T^2 = T$ and (ii)(a) holds. Now given $\mathcal{R}$-classes P and Q of M and E respectively, the set $P \times Q$ is an $\mathcal{R}$-class in $M \times E$. Conversely, every $\mathcal{R}$-class of $M \times E$ is obtained in this way. A maximal $\mathcal{R}$-class in $M \times E$ is simply the direct product of maximal

$\mathcal{R}$-classes from M and E respectively. The analogous results holds for $\mathcal{L}$-classes. Now M has a unique maximal $\mathcal{L}$-class G - the group of units of M. Similarly E has a unique maximal $\mathcal{L}$-class - the whole of E. Thus $T = M \times E$ has a unique maximal $\mathcal{L}$-class, $G \times E$. M has a unique maximal $\mathcal{R}$-class - again the group of units of M, while the maximal $\mathcal{R}$-classes of E are simply the singleton sets $\{e\}$ $(e \in E)$. So the maximal $\mathcal{R}$-classes of T, $G \times \{e\}$ $(e \in E)$, are precisely those contained in the maximal $\mathcal{L}$-class of T. Thus T satisfies (ii)(b). Finally, every $\mathcal{R}$-class of M lies below one and only one maximal $\mathcal{R}$-class, a fact that holds trivially true for E since all $\mathcal{R}$-classes are maximal. Thus every $\mathcal{R}$-class of T lies below one and only one maximal $\mathcal{R}$-class, and T satisfies (ii)(c). This completes the proof. ∎

5 Final Remarks

If we consider the left restricted wreath product then we have the following analogues to Theorems 3.9, 3.10 and 3.18 respectively:

Theorem 3.28 *Let S be an infinite semigroup whose right diagonal S-act is* not *finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}_l T$ is finitely generated if and only if*

(i) *$S^2 = S$ and $T^2 = T$;*

(ii) *S is finitely generated;*

(iii) *T has a unique maximal $\mathcal{R}$-class R, the maximal $\mathcal{L}$-classes of T being precisely those $\mathcal{L}$-classes in R.*

Corollary 3.29 *Let S be an infinite semigroup whose right diagonal S-act is not finitely generated, and let T be a finite semigroup. Then both $S\mathrm{wr}T$ and $S\mathrm{wr}_l T$*

are finitely generated if and only if $S^2 = S$, S is finitely generated and T is a monoid. ∎

Theorem 3.30 *Let S be an infinite semigroup, whose right diagonal S-act is finitely generated, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}_l T$ is finitely generated if and only if*

(i) *$S^2 = S$ and $T^2 = T$;*

(ii) *S is finitely generated.*

Theorem 3.31 *Let S be a semigroup such that S does not have finitely generated diagonal S-act, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}_l T$ is finitely presented if and only if*

(i) *S is stable;*

(ii) *$S^2 = S$;*

(iii) *$T = M \times E$ is the direct product of a monoid and a right zero semigroup.*

For monoids, we have that $A\mathrm{wr}B$ is finitely presented if and only if $A\mathrm{wr}_l B$ is finitely presented. For semigroups (in the case where T is finite) this is not the case; in fact, $S\mathrm{wr}T$ and $S\mathrm{wr}_l T$ can only both be finitely generated if T is a monoid:

Corollary 3.32 *Let S be a semigroup such that S does not have finitely generated diagonal S-act, and let T be a finite non-trivial semigroup. Then $S\mathrm{wr}_l T$ is finitely presented if and only if S is stable, $S^2 = S$ and T is a monoid.*

PROOF. If $T = M_1 \times E_1$ where M_1 is a monoid and E_1 is a left zero semigroup, and also $T = M_2 \times E_2$ where M_2 is a monoid and E_2 is a right zero semigroup, then we must have $M_1 = M_2$ and $E_1 = E_2 = \{1\}$, and hence T is a monoid. ∎

Chapter 4

Wreath Products with Infinite Top Semigroup

1 Introduction

In this chapter we look at wreath products $S\mathrm{wr}T$ where T is infinite, and consider the questions of finite generation and finite presentability. We will require the semigroup S to have an idempotent element e_S with respect to which we can define the support of functions from T into S. In what is to follow we will show that if $S\mathrm{wr}T$ is to be finitely generated, then S must be a monoid, and the idempotent e_S must be 1_S. We will also give necessary and sufficient conditions for the wreath product $S\mathrm{wr}T$ to be finitely generated.

2 Finite Generation

2.1 Idempotents of S

If T is an infinite semigroup, and S a semigroup with non-empty set of idempotents E, then we can define the wreath product $S\text{wr}T$ with respect to any given idempotent $e \in E$. If it is unclear which idempotent we are using, we will denote the wreath product $S_e\text{wr}T$. For example, if $e = 0$ then $S_0\text{wr}T = S^{(T)} \rtimes T$ - the function component of each pair (f, t) in the wreath product always has finite support - see Theorem 1.24. However, in this case the wreath product cannot be finitely generated if S is non-trivial, as we will shortly see. First we extend the concept of injectivity and introduce the notion of 'almost injectivity'.

Definition 4.1 Let T be a semigroup. An element $t \in T$ is said to be *almost (right) injective* if there exists some finite set $U \subseteq T$ such that $bt = ct$ implies $b = c$ for all $b, c \in T \setminus U$. The *deficiency* of an almost injective element t is defined to be the size of the smallest such set U. The concept of an *almost left injective* element is defined in the analogous way.

Clearly any injective element is almost injective, of deficiency 0. Trivially, any element of a finite semigroup T is almost injective. Note that if t is almost injective of deficiency m and $t' \geqslant_{\mathcal{R}} t$ then t' is also almost injective of deficiency at most m. We have the following lemma:

Lemma 4.2 *Let S and T be semigroups, where S has an idempotent element e. Let $f : T \to S$ be a function with finite support, and let t be an almost injective element of T of deficiency m. Then*

$$|\operatorname{supp}({}^{t}f)| \leqslant |\operatorname{supp}(f)| + m.$$

In particular ${}^{t}f$ also has finite support.

Proof. Let $U \subset T$ be a finite subset of T of size m, such that $bt = ct$ implies $b = c$ for $b, c \in T \setminus U$. Now $x \in \operatorname{supp}({}^{t}f)$ if and only if $xt \in \operatorname{supp}(f)$. Define $X = \{x \in T : xt \in \operatorname{supp}(f)\}$. Then $X = X_1 \cup X_2$ where $X_1 = X \cap (T \setminus U)$ and $X_2 = X \cap U$. For $x, y \in X_1 \subseteq (T \setminus U)$ we have $xt = yt$ if and only if $x = y$. Thus $|X_1| \leqslant |\operatorname{supp}(f)|$. Since $|X_2| \subseteq U$ clearly $|X_2| \leqslant |U| = m$ and so $|\operatorname{supp}({}^{t}f)| = |X| \leqslant |\operatorname{supp}(f)| + m$ as required. ■

Corollary 4.3 *Let $f : T \to S$ be a function with finite support, and let $t \in T$. If t is injective then we have $|\operatorname{supp}({}^{t}f)| \leqslant |\operatorname{supp}(f)|$. If g is invertible then we have equality: $|\operatorname{supp}({}^{g}f)| = |\operatorname{supp}(f)|$.* ■

Lemma 4.4 *Let T be an infinite semigroup and S a non-trivial semigroup with idempotent e. The wreath product $S_e\mathrm{wr}T$ is finitely generated only if $eS = S$.*

Proof. Let $\{(f_i, t_i) : i \in I\}$ be a finite generating set for $S_e\mathrm{wr}T$. We may assume that each f_i has finite support. Since T is infinite, there must exist some $y \in T$ such that $(y)f_i = e$ for all i. Given $s \in S$ write the element $(\overline{s_y}, t)$ as a product of generators:

$$(\overline{s_y}, t) = (f_1, t_1)(f_2, t_2) \dots (f_r, t_r).$$

Then $\overline{s_y} = f_1\, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$. Thus

$$s = y\overline{s_y} = yf_1(y\, {}^{t_1}f_2) \dots (y\, {}^{t_1 \dots t_{r-1}}f_r) \in eS$$

Since s was chosen arbitrarily, we have that $S = eS$ as required. ■

Lemma 4.5 *Let T be an infinite semigroup and S a non-trivial semigroup with idempotent e. If T contains an almost (right) injective element then the wreath product $S_e\mathrm{wr}T$ is finitely generated only if $Se = S$.*

PROOF. Let t be an almost injective element for T, of deficiency m say. Let $\{(f_i, t_i) : i \in I\}$ be a finite generating set for $S\mathrm{wr}T$. As before we may assume that each f_i has finite support. So we may choose $N \in \mathbb{N}$ such that

$$N > \max\{|\operatorname{supp} f_i| : i \in I\}.$$

For a contradiction, assume that there exists $z \in S \setminus Se$. Let V be a subset of T containing $N + m$ elements. We define a function $f : T \to S$ by

$$xf = \begin{cases} z & \text{if } x \in V \\ e & \text{if } x \in T \setminus V \end{cases}$$

By assumption, we can write (f, t) as a product of generators:

$$(f, t) = (f_1, t_1)(f_2, t_2) \dots (f_r, t_r).$$

Now, for all $v \in V$

$$z = (v)f = (v)f_1(vt_1)f_2 \dots (vt_1 \dots t_{r-1})f_r$$

and since $z \notin Se$ we must have $(vt_1 \dots t_{r-1})f_r \neq e$ for all $v \in V$. Thus

$$|\operatorname{supp}({}^{t_1 \dots t_{r-1}}f_r)| \geqslant |V| = N + m.$$

But $(t_1 \dots t_{r-1}) \geqslant_{\mathcal{R}} t$, and so $(t_1 \dots t_{r-1})$ is almost injective of deficiency at most m. Thus by Lemma 4.2 we have

$$|\operatorname{supp}({}^{t_1 \dots t_{r-1}}f)| \leqslant |\operatorname{supp}(f)| + m < N + m,$$

a contradiction. Thus $S = Se$ as required. ∎

Corollary 4.6 *Let T be an infinite semigroup and S a semigroup with idempotent e. If T contains an almost (right) injective element then the wreath product $S_e\mathrm{wr}T$ is finitely generated only if S is a monoid and $e = 1_S$. In particular, if T is a monoid, then $S_e\mathrm{wr}T$ is finitely generated only if S is also a monoid, and $e = 1_S$.*

Next we will give necessary and sufficient conditions on S and T for the wreath product $S_e\mathrm{wr}T$ to be finitely generated. A consequence of these conditions will be that T contains (infinitely many) almost injective elements. Thus by Corollary 4.6, $S_e\mathrm{wr}T$ can only ever be finitely generated when $e = 1_S$.

2.2 Main Theorems

We start with the following lemma, which is a combination of Theorem 2.9 and Lemma 3.12. We note that both the original results were proved in the completely general case, and so we omit the proof here.

Lemma 4.7 *Let S and T be semigroups, where T is infinite, and let e be some idempotent element of S. Then $S_e\mathrm{wr}T$ is finitely generated implies $S^2 = S$, $T^2 = T$ and that S and T are finitely generated.* ■

We note that generating sets for S and T based on a generating set for $S_e\mathrm{wr}T$ were given in Theorem 2.9.

Now we give necessary and sufficient conditions for the wreath product $S_e\mathrm{wr}T$ to be finitely generated in the case where T is infinite. We have seen that when T is finite, the conditions for finite generation of $S_e\mathrm{wr}T$ are different depending on whether S has finitely generated diagonal S-act or not. When T is infinite, the case where S is finite is no longer trivial, and so we have three different results, in Theorems 4.8, 4.10 and 4.11 below.

Theorem 4.8 *Let S be an infinite semigroup that does not have finitely generated diagonal S-act, let e be some idempotent in S, and let T be an infinite semigroup. Then $S_e\mathrm{wr}T$ is finitely generated if and only if S is a monoid, $e = 1_S$, $T^2 = T$, S and T are finitely generated and there exists a finite set $V \subseteq T$ such that for every pair $b, c \in T$ there exists a (right) injective element $t = t(b, c) \in T$ satisfying*

(i) $bt \in V$;

(ii) $c \in tT$.

PROOF. ($\Rightarrow$) By Lemma 4.7, S and T are finitely generated and satisfy $S^2 = S$ and $T^2 = T$. Since $S_e\mathrm{wr}T$ is generated by $S^{(T)} \times T$ and since it is finitely generated, it follows that there exists a finite generating set Z for $S_e\mathrm{wr}T$ which is contained in $S^{(T)} \times T$. Suppose $Z = \{(f_i, t_i) : i \in I\}$. We let $V = \bigcup_{i \in I}\{x \in T : (x)f_i \neq e\}$. Let $U = \bigcup_{i \in I}(T)f_i$. Then U and V are finite, since each f_i has finite support.

We note that since T is finitely generated, T must have a finite number of maximal $\mathcal{R}$-classes, and moreover every $\mathcal{R}$-class lies underneath a maximal one. We will show that each maximal $\mathcal{R}$-class consists entirely of injective elements. We recall that by Lemma 3.23, $\mathcal{R}$-classes preserve injectivity. Take m in some maximal $\mathcal{R}$-class R of T, and suppose that $b_1 m = b_2 m$ for $b_1, b_2 \in T$. By assumption, $S \times S$ is not finitely generated as a right S-act, so we may choose $(s_1, s_2) \in (S \times S) \setminus (U \times U)S$. Define a function $g : T \to S$ by

$$(x)g = \begin{cases} s_1 & \text{if } x = b_1 \\ s_2 & \text{if } x = b_2 \\ e & \text{otherwise,} \end{cases}$$

and write (g, m) as a product of generators:

$$(g, m) = (f_1, t_1) \ldots (f_r, t_r).$$

Then $m = t_1 \ldots t_r$ and $g = f_1\, {}^{t_1}f_2 \ldots {}^{t_1 \ldots t_{r-1}}f_r$. Now, since m is $\mathcal{R}$-maximal, we must have $t_1 \mathcal{R} m$, which in turn implies that $b_1 t_1 = b_2 t_1$, and so

$$\begin{aligned} s_1 &= (b_1)g = (b_1)f_1(b_1t_1)f_2 \ldots (b_1t_1 \ldots t_{r-1})f_r = u_1 s \\ s_2 &= (b_2)g = (b_2)f_1(b_2t_1)f_2 \ldots (b_2t_1 \ldots t_{r-1})f_r = u_2 s \end{aligned}$$

where $u_1 = (b_1)f_1$ and $u_2 = (b_2)f_1$ both lie in U. This gives us a contradiction, since $(s_1, s_2) \notin (U \times U)S$. Thus the assumption that $b_1 m = b_2 m$ was false, and therefore m is injective as required.

By Corollary 4.6, the existence of injective elements means that S must be a monoid, with identity e. It now suffices to show that for every element $b \in T$ and every $\mathcal{R}$-maximal element $c \in T$, there is an element $t \in \mathcal{R}_c$ such that $bt \in V$. Choose such elements b and c, and let $z \in S \setminus U$. Define a function $h : T \to S$ by

$$(x)h = \begin{cases} z & \text{if } x = b \\ 1_S & \text{otherwise.} \end{cases}$$

By hypothesis we can write (h, c) as a product of generators:

$$(h, c) = (f'_1, p_1) \ldots (f'_q, p_q).$$

Then

$$z = (b)h = (b)f'_1(bp_1)f'_2 \ldots (bp_1 \ldots p_{q-1})f'_q.$$

Now $(b)f'_1 \neq z$ and so there must be $2 \leqslant j \leqslant q$ such that $(bp_1 \ldots p_{j-1})f'_j \neq 1_S$, that is, $bp_1 \ldots p_{j-1} \in V$. Setting $t = (p_1 \ldots p_{j-1})$ we see that $c\mathcal{R}t$ and $bt \in V$ as required.

($\Leftarrow$) Suppose that S and T are finitely generated, that S is a monoid, $T^2 = T$ and that there exists a finite set $V \subseteq T$ satisfying the conditions stated in the theorem. We first note that condition (ii) on T means that $\mathcal{R}$-maximal elements of T must all be injective, since choosing c to be $\mathcal{R}$-maximal implies the existence of an injective t in the same $\mathcal{R}$-class, and $\mathcal{R}$-classes preserve injectivity. Let X and Y be finite generating sets for S and T respectively. Our finite generating set for $S\mathrm{wr}T$ will be

$$\mathcal{S} = \{(\overline{x_v}, y) : x \in X, y \in Y, v \in V\} \cup \{(\overline{1}, y) : y \in Y\}.$$

Since the wreath product $S\mathrm{wr}T$ is defined to be the subsemigroup of $S\mathrm{Wr}T$ generated by the elements (f, t) where $f : T \to S$ has finite support, we need only show that such generators can be written in terms of our generating set.

Let $(f, d) \in S\mathrm{wr}T$. Let m be an $\mathcal{R}$-maximal element of T such that $d \leqslant_{\mathcal{R}} m$ and let R_m be its $\mathcal{R}$-class. Since f has finite support, we may write

$$f = \overline{x^{(1)}_{b_1}} \cdots \overline{x^{(n)}_{b_n}}$$

for $x^{(i)} \in X$, $b_i \in T$. Choosing $b = b_i$ and $c = m$, by hypothesis we may find $t_i \in R_m$ and $v_i \in V$ such that $b_i t_i = v_i$. Then ${}^{t_i}\overline{x_{v_i}} = \overline{x_{b_i}}$ since t_i is injective, and so

$$(f,d) = \left(\overline{1}, t_1\right)\left(\overline{x^{(1)}_{v_1}}, t_1^{-1}t_2\right)\cdots\left(\overline{x^{(n-1)}_{v_{n-1}}}, t_{n-1}^{-1}t_n\right)\left(\overline{x^{(n)}_{v_n}}, t_n^{-1}d\right) \tag{4.1}$$

where $t_i^{-1}t_{i+1}$ is simply an element q_i such that $t_i q_i = t_{i+1}$. Such an element clearly exists if R_m contains more than 1 element. Maximal $\mathcal{R}$-classes must in fact contain an infinite number of elements when T satisfies the conditions stated, since in any given maximal $\mathcal{R}$-class there is, for each $b \in T$, an injective element t mapping b into the finite set V (by right multiplication). Obviously each t can only be used for $|V|$ distinct b's, and so there must be an infinite number of them in each $\mathcal{R}$-class. Now it is easy to see that each element of $S\mathrm{wr}T$ in the right hand side of (4.1) above can be written as a product of our generators. For example, suppose $t_i^{-1}t_{i+1} = y_1 \dots y_r$. Then

$$\left(\overline{x^{(i)}_{v_i}}, t_i^{-1}t_{i+1}\right) = \left(\overline{x^{(i)}_{v_i}}, y_1\right)(\overline{1}, y_2)\dots(\overline{1}, y_r)$$

which completes the proof. ∎

Corollary 4.9 *Let S be an infinite semigroup such that $S \times S$ is not finitely generated as a right S-act, and let T be an infinite semigroup. If $S\mathrm{wr}T$ is finitely generated, then T satisfies the following conditions:*

(i) *T has only a finite number of maximal $\mathcal{R}$-classes, each is infinite and consists entirely of injective elements.*

(ii) *There are only a finite number of minimal $\mathcal{R}$-classes.*

(iii) *Any $\mathcal{R}$-class lies between a maximal and a minimal $\mathcal{R}$-class.*

Proof.

(i) This was shown in the proof of Theorem 4.8 above.

(ii) By Theorem 4.8, condition (i), there exists a finite set $V \subseteq T$ which intersects all minimal $\mathcal{R}$-classes of T.

(iii) From Theorem 4.8, condition (i), we see that every $\mathcal{R}$-class must lie above an $\mathcal{R}$-class containing an element from the finite set V, and so all $\mathcal{R}$-classes lie above a minimal $\mathcal{R}$-class. Every $\mathcal{R}$-class must lies below a maximal $\mathcal{R}$-class, because T is finitely generated. ∎

Theorem 4.10 *Suppose that S is an infinite semigroup that has finitely generated diagonal S-act, and let e be some idempotent in S. Let T be an infinite semigroup. Then $S\mathrm{wr}T$ is finitely generated if and only if S is a monoid, $e = 1_S$, $T^2 = T$, S and T are finitely generated and there exists a finite set $V \subseteq T$ such that for every pair $b, c \in T$ there exists an almost injective element $t = t(b, c) \in T$ such that*

(i) *$bt \in V$;*

(ii) *t is injective on $T \setminus V$ and $(T \setminus V)t \wedge Vt = \emptyset$;*

(iii) *$c \in tT$.*

Proof. ($\Rightarrow$) As before, $S^2 = S$, $T^2 = T$, and S and T must be finitely generated. We may assume that $Z = \{(f_i, t_i) : i \in I\}$ generates $S\mathrm{wr}T$ where each f_i has finite support. We let $V = \bigcup_{i \in I}\{x \in T : (x)f_i \neq e\}$, and let $U = \bigcup_{i \in I}(T)f_i$. Thus U and V are finite. Again, there are only finitely many maximal $\mathcal{R}$-classes, and every $\mathcal{R}$-class lies below some maximal class. Given $b \in T \setminus V$, $c \in T$ choose m contained in some maximal $\mathcal{R}$-class, such that $c \leqslant_{\mathcal{R}} m$. Choose $b, b_1 \in T \setminus V$ such that $b \neq b_1$. Choose distinct elements s_1 and s_2 not equal to e in S. Define a function $g : T \to S$ such that

$$(x)g = \begin{cases} s_1 & \text{if } x = b \\ s_2 & \text{if } x = b_1 \\ e & \text{otherwise.} \end{cases}$$

We write (g, m) as a product of generators,

$$(g, m) = (f_1, t_1) \dots (f_r, t_r)$$

say. Then $t_1 \mathcal{R} m$, and $g = f_1 \, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$. Now if $bt_1 = b_1 t_1$ then we'd have

$$\begin{aligned} s_1 &= (b)g = (b)f_1(bt_1)f_2 \dots (bt_1 \dots t_{r-1})f_r = es = s \\ s_2 &= (b_1)g = (b_1)f_1(b_1t_1)f_2 \dots (b_1t_1 \dots t_{r-1})f_r = es = s \end{aligned}$$

since, by Lemma 4.4, e is a left identity for S (we don't yet know that it must in fact be a right identity too). This gives a contradiction, s_1 and s_2 were chosen distinct, and so $bt_1 \neq b_1 t_1$.

We claim that this t_1 is actually injective on the set $T \setminus V$. Indeed, we simply repeat the above argument, keeping c and m fixed, but varying b and b_1. In this way we can show that $bt_1 \neq b_1 t_1$ for all $b, b_1 \in T \setminus V$, since $\mathcal{R}$-classes preserve injectivity. Thus each maximal $\mathcal{R}$-class of T consists entirely of almost injective elements. Again, the existence of almost injective elements allows us to deduce that S is actually a monoid (and that $e = 1_S$).

Now given $b, c \in T$ take $z \in S \setminus U$. Define a function $h : T \to S$ by

$$(x)h = \begin{cases} z & \text{if } x = b \\ 1_S & \text{otherwise.} \end{cases}$$

Let t_1 be as before, and write (h, t_1) as a product of generators,

$$(h, t_1) = (f_1', p_1) \dots (f_q', p_q).$$

Then $z = (b)h = (b)f_1'(bp_1)f_2' \dots (bp_1 \dots p_{q-1})f_q'$. Since $(b)f_1' \neq z$ there must be $2 \leqslant j \leqslant q$ such that $bp_1 \dots p_{j-1} \in V$. We choose $p_1 \dots p_{j-1}$ to be $t = t(b, c)$.

Then clearly (i), (iii) and the first part of (ii) hold. Suppose $d \in T \setminus V$, $v \in V$ such that $dt = vt$. Choose $s \in S \setminus U$, and choose $s' \in S \setminus Us$. Define a function $h' : T \to S$ by

$$(x)h' = \begin{cases} s & \text{if } x = d \\ s' & \text{if } x = v \\ 1_S & \text{otherwise.} \end{cases}$$

Write

$$(h', t) = (f_1'', q_1) \dots (f_k'', q_k).$$

Then

$$\begin{aligned} s = (d)h' = 1_S & \ . \ (dq_1)f_2'' \dots (dq_1 \dots q_{k-1})f_k'' \\ s' = (v)h' = (v)f_1'' & \ . \ (vq_1)f_2'' \dots (vq_1 \dots q_{k-1})f_k'' \end{aligned}$$

where $q_1 \mathcal{R} t$ and so $dt = vt \Rightarrow dq_1 = vq_1 \Rightarrow s' = (v)f_1''.s$, a contradiction. So (ii) holds and we have completed one half of the proof.

($\Leftarrow$) Suppose that S and T are finitely generated, that S is a monoid, $T^2 = T$ and that there exists a finite set $V \subseteq T$ satisfying the conditions stated in the theorem. We first note that condition (ii) on T means that an $\mathcal{R}$-maximal element $t \in T$ must be almost injective, as before, and must also satisfy $Vt \wedge (T \setminus V)t = \emptyset$. Each maximal $\mathcal{R}$-class must be infinite, using the same reasoning as in Theorem 4.8. Let X and Y be finite generating sets for S and T respectively. Suppose $|V| = N$. Choose a finite subset $U \subseteq S$ such that $U \times \ldots \times U$ (N times) generates $S \times \ldots \times S$ (N times) as an S-act. Let F be the finite set of all functions $f : T \to S$ such that $(V)f \subseteq U$, and $(T \setminus V)f = 1_S$. Let m_i ($i \in I$) be a set of representatives of the maximal $\mathcal{R}$-classes. Then I is finite. For each $i \in I$, $x \in X$ define $g_{x,m_i} : T \to S$ by $(Vm_i)g_{x,m_i} = \{x\}$, $(T \setminus Vm_i)g_{x,m_i} = \{1_S\}$. Let $G = \{g_{x,m_i} : x \in X, i \in I\}$. Our finite generating set for $S\mathrm{wr}T$ will be

$$\mathcal{S} = \{(\overline{x_v}, y) : x \in X, y \in Y, v \in V\} \cup \{(f, y) : f \in F \cup G, y \in Y\} \cup \{(\overline{1}, y) : y \in Y\}.$$

Since the wreath product $S\mathrm{wr}T$ is defined to be the subsemigroup of $S\mathrm{Wr}T$ generated by the elements (f,t) where $f : T \to S$ has finite support, we need only show that such an element can be written in terms of our generating set.

Let $(f,d) \in S\mathrm{wr}T$. Let m be an $\mathcal{R}$-maximal element of T such that $m \geqslant_{\mathcal{R}} d$. Since f has finite support, we may write

$$f = \overline{x^{(1)}_{b_1}} \cdots \overline{x^{(n)}_{b_n}} \cdot f'$$

for $x^{(i)} \in X$, $b_i \in T \setminus V$, $f' : T \to S$ such that $(T \setminus V)f' = 1_S$. Note that for $m \in \{m_i : i \in I\}$, $x \in X$, we have that the function ${}^m g_{x,m} : T \to S$ maps V to x, and $T \setminus V$ to 1_S, using property (ii) of the $\mathcal{R}$-maximal elements. We may view f' as an element of $S \times \ldots \times S$ (N times) and write $f' = hk_1 \ldots k_r$ where $h \in F$, and k_i maps V to some z_i, and the rest of T to 1_S. Then $f' = h\,{}^m g_{z_1,m} \ldots {}^m g_{z_r,m}$. Now, choosing $b = b_i$ and $c = m$, by hypothesis we may find $t_i \in \mathcal{R}_m$ and $v_i \in V$ such that $b_i t_i = v_i$. Then for $i = 1, \ldots, n$, ${}^{t_i}\overline{x_{v_i}} = \overline{x_{b_i}}$ since b_i is the only element of T satisfying $b_i t_i = v_i$ by (ii). Thus we have

$$\begin{aligned}(f,d) = \Big(h, m\Big)\Big(g_{z_1,m}, m^{-1}m\Big) \;\; \cdots \;\; \Big(g_{z_r,m}, m^{-1}t_1\Big) \cdot \\ \Big(\overline{x^{(1)}_{v_1}}, t_1^{-1}t_2\Big) \;\; \cdots \;\; \Big(\overline{x^{(n-1)}_{v_{n-1}}}, t_{n-1}^{-1}t_n\Big)\Big(\overline{x^{(n)}_{v_n}}, t_n^{-1}d\Big)\end{aligned}$$

where $t_i^{-1}t_{i+1}$ is simply an element q_i such that $t_i q_i = t_{i+1}$. It is easy to see that each of the terms in the above expression can be written as products of elements from our generating set. ∎

Finally we look at the case where S is finite. The Theorem is similar to Theorem 4.10.

Theorem 4.11 *$S\mathrm{wr}T$ is finitely generated if and only if S is a monoid, $T^2 = T$, S and T are finitely generated and there exists a finite set $V \subseteq T$ such that for every pair $b \in T \setminus V, c \in T$ there exists an almost injective element $t = t(b,c) \in T$ such that*

(i) $bt \in V$

(ii) *t is injective on $T \setminus V$ and either*

(a) $Ss = S$ for all $s \in S$ **or**

(b) $(T \setminus V)t \wedge Vt = \emptyset$.

(iii) $c \in tT$

Proof. ($\Rightarrow$) As before we have that $S_e\text{wr}T$ are finitely generated implies that $S^2 = S$, $T^2 = T$, and that S and T are finitely generated. Choose a generating set $Z = \{(f_i, t_i) : i \in I\}$ of $S_e\text{wr}T$ where each f_i has finite support. Let $V = \bigcup_{i \in I}\{x \in T : xf_i \neq e\}$. Again, because T is finitely generated, there can only be finitely many $\mathcal{R}$-classes, and every $\mathcal{R}$-class must lie below some maximal $\mathcal{R}$-class. Given $b \in T \setminus V$, and $c \in T$, we choose m $\mathcal{R}$-maximal in T such that $m \geqslant_{\mathcal{R}} c$. Choose $b_1 \in T \setminus V$ such that $b_1 \neq b$, and choose distinct elements s_1 and s_2 not equal to e in S. Define a function $g : T \to S$ as follows:

$$(x)g = \begin{cases} s_1 & \text{if } x = b \\ s_2 & \text{if } x = b_1 \\ e & \text{otherwise.} \end{cases}$$

Write the element (g, m) as a product of generators:

$$(g, m) = (f_1, t_1) \dots (f_r, t_r),$$

say. Then $t_1 \mathcal{R} m$ and $g = f_1\, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$. If bt_1 were equal to b_1t_1 we'd have

$$s_1 = (b)g = (b)f_1(bt_1)f_2 \dots (bt_1 \dots t_{r-1})f_r = e.s = s,$$

$$s_2 = (b_1)g = (b_1)f_1(b_1t_1)f_2 \dots (b_1t_1 \dots t_{r-1})f_r = e.s = s,$$

since by Lemma 4.4, e is a left identity for S. This gives a contradiction, since s_1 and s_2 were chosen distinct. Thus $bt_1 \neq b_1t_1$, and so $bm \neq b_1m$. Repeating the argument with any element $b_1 \in T \setminus V$ shows that m is injective on the set

$T\backslash V$, and hence m is almost injective. The existence of almost injective elements allows us to deduce that S is a monoid, and that $e = 1_S$.

Now, $s_1 \neq 1$ and

$$s_1 = (b)f_1(bt_1)f_2 \dots (bt_1 \dots t_{r-1})f_r,$$

so there must exist $2 \leqslant j \leqslant r$ such that $bt_1 \dots t_{j-1} \in V$. Choose $t = t(b, m) = t_1 \dots t_{j-1}$. Then (i), (iii) and the first part of (ii) must hold. We note that choosing m $\mathcal{R}$-maximal will suffice for all choices $c \in T$. It remains only to complete the proof that (ii) holds.

Suppose that there exist $z_1, z_2 \in S$ such that $z_1 \notin Sz_2$ that is, (ii)(a) does not hold. We'll show that, if m is $\mathcal{R}$-maximal, then the assumption $dm = vm$ (for $d \in T \setminus V, v \in V$) leads to a contradiction, and so (ii)(b) must hold. Let m be $\mathcal{R}$-maximal, and suppose that $dm = vm$ for some $d \in T \setminus V$, $v \in V$. Define a function $h : T \to S$ as follows:

$$(x)h = \begin{cases} z_1 & \text{if } x = v \\ z_2 & \text{if } x = d \\ 1_S & \text{otherwise.} \end{cases}$$

We write

$$(h, m) = (f_1, q_1) \dots (f_k, q_k)$$

as a product of generators. Then $q_1 \mathcal{R} m$ and

$$z_1 = (v)f_1(vq_1)f_2 \dots (vq_1 \dots q_{k-1})f_k$$
$$z_2 = (d)f_1(dq_1)f_2 \dots (dq_1 \dots q_{k-1})f_k$$

Now, $(d)f_1 = 1_S$ and if $vm = dm$ then $vq_1 = dq_1$ and so $z_1 = (v)f_1 z_2 \in Sz_2$, a contradiction. This completes the direct half of the proof. ■

($\Leftarrow$) Suppose that the conditions in the Theorem hold. Let Y be a finite generating set for T, and let F be the set of all functions from T into S such that

$(T \setminus V)f = 1_S$. Since both S and V are finite, so is F. Our finite generating set for $S\mathrm{wr}T$ will be

$$\{(\overline{s_v}, y) : s \in S, v \in V, y \in Y\} \cup \{(f, y) : f \in F, y \in Y\}.$$

We need to show that an arbitrary element $(f, d) \in S\mathrm{wr}T$ (where f has finite support) can be written in terms of our generating set. Choose an $\mathcal{R}$-maximal element m such that $m \geqslant_{\mathcal{R}} d$. Since f has finite support we may write

$$f = f'\overline{s_{b_1}^{(1)}} \dots \overline{s_{b_n}^{(n)}}$$

where $f' \in F$, $s^{(i)} \in S$ and $b_i \in T \setminus V$. Now, given $b_i \in T \setminus V$ and m $\mathcal{R}$-maximal we may find $t_i \in \mathcal{R}_m$ such that $b_i t_i = v_i \in V$. Further, either we can choose all such t_i to satisfy $(T \setminus V)t_i \wedge Vt_i = \emptyset$, or we must have that $Ss = S$ for all $s \in S$.

If $(T \setminus V)t_i \wedge Vt_i = \emptyset$, then it is easy to see that

$$\overline{s_{b_i}^{(i)}} = {}^{t_i}\overline{s_{v_i}^{(i)}}.$$

If $(T \setminus V)t_i$ and Vt_i intersect then ${}^{t_i}\overline{s_{v_i}^{(i)}}$ may map some elements of V to s_i, although it is still true that b_i is the only element of $T \setminus V$ mapped to s_i. In either case, in the product

$$(\overline{1}, t_1)(\overline{s_{v_1}^{(1)}}, t_1^{-1}t_2) \dots (\overline{s_{v_{n-1}}^{(n-1)}}, t_{n-1}^{-1}t_n)(\overline{s_{v_n}^{(n)}}, t_n^{-1}d) = (g, d)$$

we have that g and our original f agree on $T \setminus V$. In the first case, we have that g maps the whole of V to the identity, in the second case, each element of S is left invertible. Either way, we may replace the initial term $(\overline{1}, t_1)$ by (h, t_1) where $h \in F$ and hg and f agree on V. Since functions from F map $T \setminus V$ to 1_S, we have

$$(f, t) = (h, t_1)(\overline{s_{v_1}^{(1)}}, t_1^{-1}t_2) \dots (\overline{s_{v_{n-1}}^{(n-1)}}, t_{n-1}^{-1}t_n)(\overline{s_{v_n}^{(n)}}, t_n^{-1}d),$$

and it is easy to check that the terms in the right hand side above are all expressible as products of our generators. ∎

3 Finite Presentability

Due to the technical nature of the finite generation results in the last section, we have no general results for when the wreath product of two semigroups S and T is finitely presented in the case where T is infinite. However, the following proposition shows how we may neatly bring together the results of the previous two chapters.

Proposition 4.12 *Let S be an infinite semigroup with idempotent e, and $T = G \times E$ be the direct product of a group and a left zero semigroup. Suppose further that $S \times S$ is not finitely generated as a right S-act. Then*

1. *$S_e\mathrm{wr}T$ is finitely generated if and only if $S^2 = S$, S and G are finitely generated, E is finite, and either G is finite or S is a monoid with identity e.*
2. *$S\mathrm{wr}T$ is finitely presented if and only if S is stable, $S^2 = S$ and both G and E are finite. (This condition does not depend on the choice of idempotent e, and so we omit it from the notation.)*

PROOF. (i) Suppose $S_e\mathrm{wr}T$ is finitely generated. By Lemma 4.7 we must have that $S^2 = S$ and that S and T are finitely generated. Thus G and E, being homomorphic images of T, are also finitely generated. In particular, E must be finite. If G is infinite, then T is also infinite, and so by Corollary 4.6 we require that S be a monoid with identity e.

Conversely, if G is infinite and S is a monoid with $e = 1_S$, then S and T are easily seen to satisfy the conditions in Theorem 4.8 (all elements of T are right injective, and we can take V to be the set $\{1_G\} \times E$). Thus $S_e\mathrm{wr}T$ is finitely generated. If G is finite, then T is finite, and S and T satisfy the conditions of 3.9, and so $S_e\mathrm{wr}T$ is finitely generated.

(ii) We will show that $S\mathrm{wr}T$ can only be finitely presented if T is finite, the result is then simply Theorem 3.18. So for a contradiction, suppose that $S\mathrm{wr}T$ is finitely presented and that T is infinite. By Corollary 4.6, S must be a monoid. As shown in (i), E must be finite, (and so by assumption, G is infinite). It is easy to see that for any $b, t \in T$, the set bt^{-1} contains exactly one element, and so we may think of the wreath product $S\mathrm{wr}T$ as the semi-direct product of $S^{(T)}$ and T (see Theorem 1.24). Let $E = \{e_1, \ldots, e_n\}$. Then

$$T = G_1 \cup G_2 \cup \ldots \cup G_n$$

is a union of groups (in fact a left zero semigroup of groups), where each $G_i = G \times \{e_i\}$ is isomorphic to G. Thus

$$S\mathrm{wr}T = S^{(T)} \rtimes T = (S^{(T)} \rtimes G_1) \cup (S^{(T)} \rtimes G_2) \cup \ldots \cup (S^{(T)} \rtimes G_n)$$

is a left zero semigroup of groups. By [1, Corollary 5.6] we know that if $S\mathrm{wr}T$ is finitely presented, then $S^{(T)} \rtimes G_1 \cong S^{(T)} \rtimes G$ is also finitely presented.

Define a map $\phi : S^{(T)} \rtimes G \to S\mathrm{wr}G$ by

$$(f, g)\phi = (f{\restriction_G}, g).$$

As in Proposition 2.14, ϕ is an epimorphism, and the congruence $\ker\phi$ is finitely generated. Thus $S\mathrm{wr}G$, as the quotient of the finitely presented monoid $S^{(T)} \rtimes G$ by the finitely generated congruence $\ker\phi$, is finitely presented. Using Theorem 2.3, we see that $S\mathrm{wr}G$ can only be finitely generated if G is finite, and so $T = G \times E$ must also be finite - a contradiction. Thus T cannot be infinite if $S\mathrm{wr}T$ is finitely presented, and so (ii) is proved. ∎

Chapter 5

Further Properties of Wreath Products

1 Introduction

In this Chapter we look at some other properties of the wreath product of two semigroups. In Section 2 we attempt to give a description of the elements of $S\mathrm{wr}T$, before investigating other finiteness conditions in Section 3. Finally in Section 4 we look at some sample presentations for 'awkward' wreath products, namely wreath products where finite support is not preserved in function components.

2 Elements of $S_e\mathrm{wr}T$

The wreath product of two groups G and H can be described as the set of all pairs (f, h) where f is a function from H to G with finite support, and h is an element of H. Unfortunately it is generally more difficult to describe the elements of the wreath product of two semigroups S and T. We still have pairs (f, t) where

f is a function from T into S, but as we saw in Chapter 1, f need not have finite support. In this section we will make some observations about elements of the general wreath product $S_e\text{wr}T$ which will be useful later on.

We saw before that provided the sets bt^{-1} $(b, t \in T)$ are all finite, the function component f of any element (f, t) of $S_e\text{wr}T$ will have finite support. This is not the case for an infinite semigroup T in general. Given an element $t \in T$ we may consider the partition P_t of T defined by the equivalence relation $\sim$, where for two elements x and y in T we have $x \sim y$ if and only if $xt = yt$. We let $b = xt$ label the equivalence class containing x. The two extreme examples of this are given below:

Example 5.1 Let G_0 be the infinite cyclic group with a multiplicative zero attached. Then

- P_1 partitions T into singleton classes. The class containing x is labelled simply by x.
- P_0 partitions T into a single class, labelled 0.

Since P_t is an equivalence relation, we may write xP_ty whenever x and y lie in the same equivalence class. We define a partial order on the partitions P_t. For $t_1, t_2 \in T$ we say $P_{t_1} \geqslant P_{t_2}$ if whenever $xP_{t_1}y$ we also have $xP_{t_2}y$. It is easy to see that if $t_1 \geqslant_{\mathcal{R}} t_2$ then $P_{t_1} \geqslant P_{t_2}$. The converse fails, for example, in a zero semigroup.

As noted in Chapter 1, we may write any element (f, t) of $S_{e_S}\text{wr}T$ as

$$(f, t) = (f_1, t_1) \ldots (f_r, t_r), \tag{5.1}$$

where each f_i does have finite support. Then

$$f = f_1\,{}^{t_1}f_2 \ldots {}^{t_1 \ldots t_{r-1}}f_r. \tag{5.2}$$

Since $x\,{}^t\!f = (xt)f$ we see that ${}^t\!f$ is constant on equivalence classes of P_t. By convention, even if T does not have an identity we will write ${}^1\!f$ to denote f and P_1 to denote the partition of T into singleton classes. Trivially any function ${}^1\!f$ is constant on equivalence classes of P_1. Suppose f has finite support. We completely determine ${}^t\!f$ by specifying an element of S for ${}^t\!f$ to map each equivalence class of P_t to, since ${}^t\!f$ is constant on classes. In fact, if f has finite support then $z\,{}^t\!f \neq e_S$ only if z is P_t equivalent to some element in the support of f. So although P_t may divide T up into infinitely many equivalence classes, we need only specify an element of S for finitely many of them in order to determine ${}^t\!f$ - the rest must automatically map to e_S. With this representation for two functions ${}^{t_1}\!f$ and ${}^{t_2}\!g$ say, we obtain a representation for their product ${}^{t_1}\!f\,{}^{t_2}\!g$ by taking the intersection of the partitions P_{t_1} and P_{t_2}. If ${}^{t_1}\!f$ maps the equivalence class of P_{t_1} containing x to s_1 and ${}^{t_2}\!g$ maps the equivalence class of P_{t_2} containing x to s_2 then their product maps the intersection of these two classes to $s_1 s_2$.

We note that the T-component of an element $(f,t) \in S\mathrm{wr}T$ plays a large role in defining what form f may have. In general we have $f = f_1\,{}^{t_1}\!f_2 \ldots {}^{t_1 \ldots t_{r-1}}\!f_r$ where each f_i has finite support, and $t_1 \ldots t_{r-1} \geqslant_{\mathcal{R}} t$. If f is to have infinite support then there must exist some j with $2 \leqslant j \leqslant r$ such that ${}^{t_1 \ldots t_{j-1}}\!f_j$ has infinite support. Thus $t_1 \ldots t_{j-1}$ must map (by right multiplication) infinitely many elements of T into a single element, and so $t_1 \ldots t_{j-1}$, and hence t, cannot be (almost) injective.

While these observations do not really give us the useful general description of elements that we have in the group case, we can use them to describe specific examples. In what follows we will assume that $S = \mathbb{Z}$, the infinite cyclic group, and so $e_S = 0$, the additive identity. Since we are using 0 as an identity element, to avoid confusion we will denote a multiplicative zero by z.

Example 5.2 If T is a semigroup such that the sets bt^{-1} are finite for all $b, t \in T$

(e.g. groups, cancellative semigroups, finite semigroups) then each equivalence relation P_t partitions T into finite equivalence classes. Thus ${}^t f$ has finite support whenever f does, and so as in the group case, elements of $S\mathrm{wr}T$ are precisely pairs (f,t) where $f : T \to S$ has finite support, and $t \in T$.

Example 5.3 Let T be an infinite group with a multiplicative zero z adjoined. The set bt^{-1} is finite unless both b and t are equal to z, and P_t partitions T into equivalence classes of size one unless $t = z$. Acting on a function f with finite support by any element other than z will preserve the finiteness of support (in fact invertible elements preserve the size of the support, not just finiteness). If $t \neq z$ then since $z \not\geqslant_{\mathcal{R}} t$ the function component of any element (f,t) of $S\mathrm{wr}T$ must have finite support. On the other hand, if $t = z$ and $g : T \to S$ maps z to s, then ${}^z g$ is the constant function with image $\{s\}$. The elements of $S\mathrm{wr}T$ are as follows:

- (f,t) where f has finite support.
- (f,z) where f maps all but finitely many elements to a single element s. In particular, f need not have finite support.

Example 5.4 Let $\mathbb{Z}^z$ denote an infinite cyclic group with multiplicative zero 'z' attached (for ease of notation we use addition as the semigroup operation, and specify that $i+z = z+i = z+z = z$ for all $i \in \mathbb{Z}$). Let $T = \mathbb{Z}^z \times \mathbb{Z}_n$. The only non-invertible elements of T are (z,j) for $0 \leqslant j \leqslant n-1$. Now $P_{(z,j)}$ partitions T into n infinite equivalence classes, D_i $(0 \leqslant i \leqslant n-1)$, where $D_i = \{(m,i) : m \in \mathbb{Z}^z\}$. These are the only P_t $(t \in T)$ which give infinite equivalence classes, and we note that the elements $\{(z,j) : 0 \leqslant j \leqslant n-1\}$ are $\mathcal{R}$-equivalent. Thus $(f,t) \in S\mathrm{Wr}T$ is an element of $S\mathrm{wr}T$ if and only if f is 'almost constant' on each equivalence class D_i, (here f is almost constant on D if there exist a finite subset D' of D such that f is constant on $D \setminus D'$) and further, either f has finite support or $t = (z,j)$ for some j.

2.1 Units in $S\mathrm{wr}T$

Next we look at the group of units of an arbitrary wreath product. $S_e\mathrm{wr}T$. First we give a necessary and sufficient condition for the wreath product to be a monoid and actually have a group of units. We will use the following fact about direct products: an element (s,t) is invertible in $S \times T$ if and only if s is invertible in S and t is invertible in T.

Lemma 5.5 *Let S and T be semigroups. The wreath product $S_{e_S}\mathrm{wr}T$ is a monoid if and only if both S and T are monoids, and either T is finite or $e_S = 1$.*

PROOF. Suppose first that S and T are monoids. Let $h : T \to S$ be defined by $xh = 1$ for all $x \in T$. The function h has finite support if either T is finite, or $e_S = 1$, and so $(h, 1)$ is an element of $S_{e_S}\mathrm{wr}T$. It is simple to check that $(h, 1)$ is in fact an identity for $S_{e_S}\mathrm{wr}T$. Conversely, if (f, t) is an identity for $S_{e_S}\mathrm{wr}T$ then t must be an identity for T, and so f must be an identity for $S^{(T)}$. Since $S^{(T)}$ is simply the direct product of copies of S, S must itself have an identity element and f must map the whole of T to 1_S. This is only possible if either T is finite or $e_S = 1$. ■

Our next proposition specifies the group of units of $S\mathrm{wr}T$. This is not dependent on the idempotent e_S with respect to which the wreath product is defined, providing the conditions in Lemma 5.5 are satisfied.

Proposition 5.6 *Let S and T be monoids, with group of units G and H respectively and suppose that $S\mathrm{wr}T$ is a monoid. Then the group of units of $S\mathrm{wr}T$ is simply $G^{(T)} \rtimes H$.*

PROOF. We have omitted the idempotent e_S from our notation for the wreath product because it does not affect the proof. We only use the fact that $\bar{1}$ must be an element of $S^{(T)}$ if $S\mathrm{wr}T$ is to be a monoid (a consequence of Lemma 5.5 above). Let $f \in G^{(T)}$, that is, f is a function from T into G with finite support. Using Lemma 4.2 we see that for any $h \in H$, ${}^{h}f$ also has finite support (in fact $|\operatorname{supp}({}^{h}f)| = |\operatorname{supp}(f)|$), since invertible elements are injective. Thus the semi-direct product $G^{(T)} \rtimes H$ is well defined. Take $(f, h) \in G^{(T)} \rtimes H$. Clearly any element $f \in G^{(T)}$ has an inverse f^{-1} in $G^{(T)}$, and then it is easy to see that $(\overline{1}, h^{-1})(f^{-1}, 1)$ is an inverse for (f, h).

Conversely, if (f, t) an invertible element of $S\mathrm{wr}T$ then t must be an invertible element of T, and so $t \in H$. Then $(f, t)(\overline{1}, t^{-1}) = (f, 1)$ must be invertible, since $(\overline{1}, t^{-1})$ is obviously invertible. Thus f must lie in $G^{(T)}$ as required. ∎

3 Other Finiteness Conditions

In this section we will show that the finiteness conditions *periodicity* and *local finiteness* are preserved by taking wreath products. We start by defining these terms.

Definition 5.7 Let S be a semigroup. We say

(i) S is *periodic* if each element has finite order, that is, the subsemigroup generated by any element in S is finite.

(ii) S is *locally finite* if all finitely generated subsemigroups of S are finite.

It is clear that a locally finite semigroups is also periodic. We also note that a finitely generated, locally finite semigroup is necessarily trivial. A finitely generated periodic semigroup however need not be finite - see [21, Chapter 14] for example. Before proving the main theorems, we make the following simple observations.

Lemma 5.8 *Let S be a semigroup, and let $s \in S$ be an element. Then the following are equivalent:*

(i) *s has finite order;*

(ii) *There exist $1 \leqslant m < n$ such that $s^m = s^n$.*

(iii) *There exists $1 \leqslant n$ such that s^n is idempotent.* ∎

Lemma 5.9 *Let S be a semigroup, and let U be a finite subset of S containing only elements of finite order. Then we may find integers $1 \leqslant m < n$ such that $u^m = u^n$ for all $u \in U$.*

PROOF. Using Lemma 5.8, for each $u_i \in U$ we may choose integers $1 \leqslant m_i < n_i$ such that $u_i^{m_i} = u_i^{n_i}$. It is then easily verified that $m = \max\{m_i\}$ and $n = m + \operatorname{lcm}\{n_i - m_i\}$ satisfy the required condition. ∎

We will automatically assume that when T is infinite, S has a suitable idempotent element e_S with which to define the wreath product $S_{e_S}\mathrm{wr}T$. The choice of idempotent does not affect either of the following proofs, and so we omit it from the notation.

Theorem 5.10 *Let S and T be non-trivial semigroups. The wreath product $S\mathrm{wr}T$ is periodic if and only if both S and T are periodic.*

PROOF. First we assume that $S\mathrm{wr}T$ is periodic. Then T, being a homomorphic image, is also periodic. Thus we may choose an idempotent element e of T. Given $s \in S$ we choose some $f \in S^{(T)}$ such that $ef = s$. By assumption, the element (f, e) has finite order, and so there exist $m < n \in \mathbb{N}$ such that $(f, e)^m = (f, e)^n$. Thus

$$(f\,{}^{e}f\,{}^{e^2}f \dots {}^{e^{m-1}}f, e^m) = (f\,{}^{e}f\,{}^{e^2}f \dots {}^{e^{n-1}}f, e^n)$$

and so equating first components we have $f({}^{e}f)^{m-1} = f({}^{e}f)^{n-1}$. Evaluating at $e \in T$ we see that $s^m = s^n$, and so s has finite order. Since s was arbitrary, S must be periodic as required.

Now we assume that both S and T are periodic. Let (f, t) be an arbitrary element in $S\mathrm{wr}T$. As noted in Chapter 1, f need not have finite support, but does have finite image. For any semigroup element x and positive integer p, x has finite order if and only if x^p has finite order. Therefore, since T is periodic, we may assume that $t = e$ is idempotent. Let $U = (T)f$ be the image of f. Since U is a finite set of periodic elements, using Lemma 5.9 we may find positive integers $m < n$ such that $u^m = u^n$ for all $u \in U$. Thus for all $z \in T$ we have $(ze)f \in U$ and so

$$z(f({}^{e}f)^m) = zf\big((ze)f\big)^m = zf\big((ze)f\big)^n = z(f({}^{e}f)^n).$$

Thus $f({}^{e}f)^m = f({}^{e}f)^n$ and so $(f, e)^{m+1} = (f, e)^{n+1}$ as required. ■

Theorem 5.11 *Let S and T be semigroups. The wreath product $S_{e_S}\mathrm{wr}T$ is locally finite if and only if both S and T are locally finite.*

Proof. Suppose $S\mathrm{wr}T$ is locally finite. Then T, being a homomorphic image of $S\mathrm{wr}T$, is also locally finite. Thus T is periodic, and we may choose $c \in T$ such that c is idempotent. Given a finite subset of S, $\{s_i : i \in I\}$, we let g_i $(1 \leqslant i \leqslant n)$ be elements of $S^{(T)}$ mapping c to s_i respectively. By hypothesis, the semigroup

$$\big\langle\{(g_i, c) : i = 1 \ldots n\}\big\rangle$$

is finite. For any $s \in \langle\{s_i : i \in I\}\rangle$ we can write $s = s_1 \ldots s_r$ say. So

$$(c)g_1\,{}^{c}g_2 \ldots {}^{c^{r-1}}g_r = s_1 \ldots s_r = s.$$

Consequently we must have that $\langle\{s_i : i \in I\}\rangle$ is finite since $\langle\{(g_i, c) : i \in I\}\rangle$ is finite and can therefore only give a finite number of elements of S when an element's function component is evaluated at c. So S is locally finite.

Now suppose S and T are locally finite. Given elements $(f_i, t_i) \in S\mathrm{wr}T$ for i in some finite set I we show that there are only a finite number of choices for $(f, t) \in \langle (f_i, t_i) \rangle$. Assume $(f, t) = (f_1, t_1) \dots (f_r, t_r)$. Then $t = t_1 t_2 \dots t_r$ and so t must come from $\langle \{t_i : i \in I\} \rangle$ which is finite by hypothesis. Now

$$f = f_1 \, {}^{t_1}f_2 \dots {}^{t_1 \dots t_{r-1}}f_r$$

and $(T)({}^{t}f_i) = (Tt)f_i \subseteq (T)f_i$ which is finite. So $(T)f \subseteq \langle \bigcup_{i \in I} (T)f_i \rangle$ which is finite, since each $(T)f_i$ is finite and S is locally finite.

This function f is the product of functions f_j and f_k^z for $j, k \in I$ and $z \in \langle \{t_i : i \in I\} \rangle$. Each such function g defines a finite partition $B_1, \dots B_m$ of T, such that g is constant on each B_l. Given that there are only a finite number of such functions to consider, we may find a finite partition $C_1, C_2 \dots, C_n$ where all such functions are constant on each C_l. Then any product f of functions will also be constant on each C_l. This gives us an upper bound for the number of choices for f, since $(T)f$ is finite.

Thus $\langle (f_i, t_i) \rangle$ is finite as required, and $S\mathrm{wr}T$ is locally finite. ∎

4 Sample Presentations for Monoids

In Theorem 2.13 we saw a presentation for the wreath product of two monoids. Unfortunately this presentation is only valid when all the sets bt^{-1} $(b, t \in T)$ are finite, that is, when all the partitions P_t $(t \in T)$ give exclusively finite equivalence classes. A presentation for the more general case is hard to find, here we content ourselves with a couple of specific example presentations, corresponding to Examples 5.3 and 5.4 in Section 2.

Example 5.12 Let $S = \langle a, a^{-1} \,|\, R_S \rangle$ and let $T = \langle g, g^{-1}, z \,|\, R_T \rangle$ be monoid presentations where

$$\begin{aligned} R_S &= \{aa^{-1} = a^{-1}a = 1\} \\ R_T &= \{gg^{-1} = g^{-1}g = 1, z^2 = zg = gz = z\}. \end{aligned}$$

Then S is the infinite cyclic group, and T is the infinite cyclic group with a multiplicative zero (z) adjoined. As observed in Example 5.3, the sets bt^{-1} are not all finite, and so the presentation for $S\text{wr}T$ given in 2.13 would contain words of infinite length. We claim that the monoid presentation

$$\mathcal{P} = \langle X \mid R \rangle = \langle \overline{a_1}, \overline{a_1}^{-1}, \overline{a_z}, \overline{a_z}^{-1}, g, g^{-1}, z \mid R_{S'}, R_T, R_C, R_1, R_2, R_3 \rangle$$

defines $S\text{wr}T$, where $g^{-1}, \overline{a_1}^{-1}$ and $\overline{a_z}^{-1}$ are simply formal symbols, and we allow notation such as $g^{-i} = (g^{-1})^i$, $x^0 = 1$, and ${}^{g^i}\overline{a_1} = g^i\overline{a_1}g^{-i}$. The sets of relations are defined as follows:

$$\begin{aligned} R_{S'} &= \{\overline{a_1}\,\overline{a_1}^{-1} = \overline{a_1}^{-1}\overline{a_1} = 1 = \overline{a_z}\,\overline{a_z}^{-1} = \overline{a_z}^{-1}\overline{a_z}\} \\ R_T &= \{gg^{-1} = g^{-1}g = 1, z^2 = zg = gz = z\} \\ R_C &= \{\text{The following elements commute: } \overline{a_z}, \overline{a_z}^{-1},\ {}^{g^i}\overline{a_1},\ {}^{g^i}(\overline{a_1}^{-1})\ (i \in \mathbb{Z})\} \\ R_1 &= \{\text{The following elements commute: } g, g^{-1}, \overline{a_z}, \overline{a_z}^{-1}\} \\ R_2 &= \{z\overline{a_1} = z = z\overline{a_1}^{-1}\} \\ R_3 &= \{z\overline{a_z}^{\,i}z = z\overline{a_z}^{\,i}\ (i \in \mathbb{Z})\} \end{aligned}$$

Let M be the monoid defined by $\mathcal{P}$. Define $\phi : M \to S\text{wr}T$ in the expected way: ϕ maps generators $\overline{a_1}$ and $\overline{a_z}$ to $(\overline{a_1}, 1)$ and $(\overline{a_z}, 1)$ respectively, $\overline{a_1}^{-1}$ and $\overline{a_z}^{-1}$ to $\left(\overline{(a^{-1})_1}, 1\right)$ and $\left(\overline{(a^{-1})_z}, 1\right)$ respectively, and g, g^{-1} and z to $(\overline{1}, g)$, $(\overline{1}, g^{-1})$ and $(\overline{1}, z)$ respectively. We then extend ϕ to a homomorphism in the usual way. To show that M and $S\text{wr}T$ are isomorphic we need to show that ϕ respects the relations R (is well defined), and that it is bijective. It is clear that ϕ respects $R_{S'}$ and $R_{T'}$ while R_C simply says that elements of $S^{(T)}$ commute. Now since ${}^{g}\overline{a_z} = \overline{a_z}$ as functions from T into S (with the usual action from T) we have

$$(\overline{1}, g)(\overline{a_z}, 1) = ({}^{g}\overline{a_z}, g) = (\overline{a_z}, g) = (\overline{a_z}, 1)(\overline{1}, g)$$

and so $g\phi$ and $\overline{a_z}\phi$ commute. In the same way we can show that ϕ respects all the relations from R_1. Similarly, since $tz \neq 1$ for all $t \in T$ we have ${}^{z}\overline{a_1} = \overline{1}$, and so

$$(\overline{1}, z)(\overline{a_1}, 1) = (\overline{1}, z).$$

It is easy to check that the other relation in R_2 is also preserved by ϕ. Finally

$$(\overline{1}, z)(\overline{a_z}^i, 1)(\overline{1}, z) = ({}^{z}\overline{a_z}^i, z) = (\overline{1}, z)(\overline{a_z}^i, 1),$$

and so the relations R_3 are preserved by ϕ.

Now, $T = \{z, 1\}\mathbb{Z}$, and so ϕ maps onto the generating set for $S\text{wr}T$ given in 2.8. Thus ϕ is onto, and it remains only to show that ϕ is one-one. Let $X' = X \setminus \{z, \overline{a_z}, \overline{a_z}^{-1}\}$, and let R' be the subset of relations $(u, v) \in R$ such that $u, v \in (X')^*$. The presentation $\mathcal{P}' = \langle X' \mid R' \rangle$ is easily seen to be equivalent to the one given in [13, Corollary 2.3] and so defines $\mathbb{Z}\text{wr}\mathbb{Z}$.

We claim that, given a word in $w \in X^*$, we may write, using the relations in R, $w = w_1 w_2$, where $w_1 \in (X')^*$, $w_2 \in \{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$. If z is not a sub-word of w, we may move any occurrences of $\overline{a_z}$ and $\overline{a_z}^{-1}$ to the right of generators in X' using relations from R_C and R_1 and we are done. Otherwise, suppose that $w \equiv \alpha z \beta$, where z is not a sub-word of α. Unless β contains an element from X' we are done, so assume the word β contains $n \geqslant 1$ symbols from X', and let y be the first such. The symbol y' immediately preceding y in the word w must be an element of $\{z, \overline{a_z}, \overline{a_z}^{-1}\}$. If $y' = z$, then we may eliminate y using the relation $zy = z$ from either R_T or R_2. Otherwise y commutes with y', using relations from R_1 and R_C. In any event, we can use commuting relations to force y adjacent (to the right) to an occurrence of z, and then eliminate y, reducing the value of n by one. Since we only use either commuting relations or relations that remove elements of X' from w, our claim follows by induction.

It is easy to see that we may use relations from R_3 to write w_2, a word in

$\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$, as

$$\overline{a_z}^i(z^\epsilon)\overline{a_z}^j \tag{5.3}$$

for $i, j \in \mathbb{Z}$, $\epsilon \in \{0, 1\}$. Let $p = p_1p_2$ and $q = q_1q_2$ be two words in $\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*(X')^*$ such that $p\phi = q\phi$. We may assume that p_2 and q_2 have the same form as (5.3). We write $p_1\phi = (f, c)$, $q_1\phi = (f', c')$, $p_2\phi = (g, d)$, and $q_2\phi = (g', d')$. Since no z occurs in either p_1 or q_1, both f and f' must have finite support, and neither c nor c' may equal z. Furthermore, since no $\overline{a_z}^{\pm 1}$ occur either, both f and f' must map z to 1. Since both f and f' have finite support (as yet we cannot say the supports are equal as sets), we may choose $b \in T \setminus \{z\}$ outside the support of both. Write $p_2 = \overline{a_z}^{i_p}(z^{\epsilon_p})\overline{a_z}^{j_p}$ and $q_2 = \overline{a_z}^{i_q}(z^{\epsilon_q})\overline{a_z}^{j_q}$, where $i_p, j_p, i_q, j_q \in \mathbb{Z}$ and $\epsilon_p, \epsilon_q \in \{0, 1\}$. Obviously $p\phi$ has T-component z if and only if $\epsilon_p = 1$, while $q\phi$ has T-component z if and only if $\epsilon_q = 1$. Thus $\epsilon_p = \epsilon_q$. It is easy to check that g maps z to $a^{i_p+j_p}$, and everything else (in particular, b) to $a^{\epsilon_p * j_p}$. Similarly g' maps z to $a^{i_q+j_q}$, and everything else to $a^{\epsilon_q * j_q}$. Now, the function component of $p\phi$, f_p say, is equal to $f({}^c g)$, while the function component f_q of $q\phi$ is equal to $f'({}^{c'} g')$. Then

$$\begin{aligned} zf_p &= zf.(zc)g = 1.zg = zg \\ zf_q &= zf'.(zc')g' = zg' \end{aligned}$$

and so g and g' must agree on z. That is, $i_p + j_p = i_q + j_q$. Also

$$\begin{aligned} bf_p &= bf.(bc)g = 1.(bc)g = (bc)g \\ bf_q &= bf'.(bc')g' = (bc')g'. \end{aligned}$$

Since neither b, c nor c' is equal to z, neither are either bc or bc'. Thus $(bc)g = a^{\epsilon_p * j_p}$ while $(bc')g' = a^{\epsilon_q * j_q}$. Hence $j_p = j_q$ and so $i_p = i_q$. Thus p_2 and q_2 (in the form given in (5.3)) are identical as words in $\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$, and we have $g = g'$. Also, g is constant on $T \setminus z$, (and $c \neq z$) and so $x\,{}^c g = (xc)g = cg$ for all $x \in T$.

Thus $g = {}^{c}g$. Similarly $g' = {}^{c'}g'$. Then, since we must have $f_p = f_q$,

$$fg = f\,{}^{c}g = f_p = f_q = f'\,{}^{c'}g' = f'g' = f'g$$

and since all functions from T into S are invertible, $f = f'$.

We were given two words, p and q in X^* such that $p\phi = q\phi$. We used relations from R to write $p = p_1p_2$ and $q = q_1q_2$ as elements of $(X')^*\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$ with p_2 and q_2 in the same form as (5.3). We showed p_2 and q_2 must in fact be identical words in $\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$. We now show that we may transform p_1 to q_1 using relations from R. Indeed, $p_1\phi$ and $q_1\phi$ are elements of the subgroup $\mathbb{Z}\mathrm{wr}\mathbb{Z}$, whose defining relations are all found in $R'\phi$. Now $p_1\phi = (f, c)$ and $p_2\phi = (f', c')$ where $f = f'$. Necessarily we have $cd = c'd'$, and so if $d = d'$ is not equal to z, we must have $c = c'$, and we are done, simply using the same relations to transform p_1 into p_2 as we would do in the group ($\mathbb{Z}\mathrm{wr}\mathbb{Z}$) case. If $d = d' = z$ then since $(f, c) = (f, 1)(\overline{1}, c)$ we may (using the group relations) transform p_1 into $\gamma\delta$ where $\gamma\phi = (f, 1)$ and $\delta\phi = (\overline{1}, c)$. Similarly we transform q_1 into $\gamma\delta'$ where again $\gamma\phi = (f, 1)$ but $\delta'\phi = (\overline{1}, c')$. We choose δ and δ' as words in $\{g, g^{-1}\}^*$. Then $p = \gamma\delta p_2$ and $q = \gamma\delta' q_2$, where p_2 and q_2 are identical words in $\{z, \overline{a_z}, \overline{a_z}^{-1}\}^*$. It remains only to use the commuting relations in R_1 to move δ and δ' towards the z in p_2 (which must exist in the case $d = z$), and then cancel using $gz = z$.

We have shown that if p and q are words in X^* such that $p\phi = q\phi$ then p may be transformed to q using the relations in R. Thus ϕ is one-one, and the presentation $\mathcal{P} = \langle X \mid R \rangle$ defines $S\mathrm{wr}T$ as claimed. ■

Example 5.13 As in the last example, we let S be the infinite cyclic group $\mathbb{Z}$. We denote by $\mathbb{Z}^z$ the infinite cyclic group with a multiplicative zero adjoined, and let $T = \mathbb{Z}^z \times \mathbb{Z}_n$ ($n \geqslant 2$). Then $S = \langle a, a^{-1} \mid R_S \rangle$ and $T = \langle g, g^{-1}, h, z \mid R_T \rangle$

where

$$\begin{aligned} R_S &= \{aa^{-1} = 1 = a^{-1}a\}, \\ R_T &= \{gg^{-1} = g^{-1}g = h^n = 1, gh = hg, zh = hz, gz = zg = z^2 = z\}. \end{aligned}$$

Then the presentation

$$\mathcal{P} = \langle X \mid R \rangle = \langle \overline{a_i}^{\pm 1}\ (0 \leqslant i \leqslant n), g^{\pm 1}, h, z \mid R_{S'}, R_T, R_C, R_1, R_2, R_3, R_4 \rangle$$

defines $S\mathrm{wr}T$ (using the same notation as in the last example). The sets of relations are defined as follows:

$$\begin{aligned} R_{S'} &= \{\overline{a_i}^{-1}\overline{a_i} = \overline{a_i}\,\overline{a_i}^{-1} = 1\ (0 \leqslant i \leqslant n)\} \\ R_T &= \{gg^{-1} = g^{-1}g = h^n = 1, gh = hg, zh = hz, gz = zg = z^2 = z\} \\ R_C &= \{\text{The following commute: } {}^{t}\overline{a_i}^{\pm 1}\ (0 \leqslant i \leqslant n, t \in \langle g, h \rangle)\} \\ R_1 &= \{\text{The following commute: } g, g^{-1}, \overline{a_i}^{\pm 1}(1 \leqslant i \leqslant n)\} \\ R_2 &= \{z\overline{a_0} = z = z\overline{a_0}^{-1}\} \\ R_3 &= \{z\overline{a_i}^{j}z = z\overline{a_i}^{j}\ (1 \leqslant i \leqslant n, j \in \mathbb{Z})\} \\ R_4 &= \{{}^{h^i}\overline{a_n} = \overline{a_{n-i}},\ (0 \leqslant i \leqslant n-1)\} \end{aligned}$$

Obviously the generators $\overline{a_i}^{\pm}$ for $1 \leqslant i \leqslant n-1$ are redundant, using the relations from R_4, but they make the presentation slightly simpler. The proof that this presentation does in fact define $S\mathrm{wr}T$ is very similar to that in Example 5.12, and is omitted. ∎

Chapter 6

Diagonal Acts

1 Introduction

The notion of a diagonal act was briefly introduced in Chapter 3. Now we give a more rigorous definition.

Definition 6.1 Let S be a semigroup and let X be a set.

(i) We say that X is a *right S-act* if there is an action $(x, s) \mapsto xs$ from $X \times S$ into X with the property that $x(st) = (xs)t$ for all $x \in X$, $s, t \in S$. We define the notion of a left S-act analogously.

(ii) We say X is a *bi S-act* if it is both a right and a left S-act and these actions are linked by

$$s(xt) = (sx)t \quad (s, t \in S,\ x \in X).$$

A right S-act X is generated by a subset $U \subseteq X$ if $US^1 = X$. (As usual, S^1 stands for S with an identity adjoined if S does not already have one.) Similarly a left S-act X is generated by a subset $U \subseteq X$ if $S^1U = X$. A bi S-act X is generated by a subset $U \subseteq X$ if $S^1US^1 = X$. If S is a monoid then we may replace S^1 by S in these definitions. Clearly a bi S-act that is finitely generated as a right or left S-act is also finitely generated as a bi S-act.

Example 6.2 Any semigroup S is a right S-act with action $(s_1, s_2) \mapsto s_1 s_2$. Obviously S is finitely generated as a right S-act if and only if S is finitely generated as a semigroup.

Definition 6.3 For any semigroup S, the set $S \times S$ can be made into a right, left or bi S-act by defining

$$(x, y)s = (xs, ys)$$
$$s(x, y) = (sx, sy)$$

for all $x, y, s \in S$; we refer to these acts as the *diagonal right, left,* and *bi S-acts* respectively.

The following lemma, which we will use frequently throughout the chapter, allows us to choose a more simple generating set for $S \times S$.

Lemma 6.4 *If $S \times S$ is finitely generated as a right, left or bi S-act, then we may assume that the generating set is of the form $U \times U$ for some finite subset U of S.*

Proof. If $S \times S$ is generated by a finite set $Y \subseteq S \times S$ then take

$$U = \{s \in S : (s, t) \in Y \text{ or } (t, s) \in Y \text{ for some } t \in S\}.$$

Then U must be finite and $U \times U$ finitely generates $S \times S$ as required. The left and bi act cases are proved in the same way. ■

The initial motivation for this area of study was to help with the investigation of wreath products of semigroups. In Chapter 3, we referred to finitely generated semigroups whose right diagonal S-acts are also finitely generated. That we were justified in doing this is demonstrated in Section 2 below, where we show that such semigroups do in fact exist. In Section 3 we prove various general results about these semigroups, including the results required in Chapter 3. In Section 4 we look at further examples, and in particular show the independence of properties of $S \times S$ as a right S-act from those as a left or bi S-act. Finally, in Section 5 we investigate the connection between diagonal acts and power semigroups. First however we look at some examples.

Example 6.5 Let $\mathbb{N}$ be the monoid of integers under addition. Then $\mathbb{N}$ does not have finitely generated diagonal right act.

Proof. Suppose that $\mathbb{N} \times \mathbb{N} = (U \times U) + \mathbb{N}$ for some finite subset $U \subseteq \mathbb{N}$. (We need not use $\mathbb{N}^1$ since $\mathbb{N}$ is already a monoid.) Choose $N > \max\{u : u \in U\}$. Note that for any $(a, b) \in (U \times U) + \mathbb{N}$ we have $(a, b) = (u+n, v+n)$ for $u, v \in U$, $n \in \mathbb{N}$, and so

$$a - b = (u + n) - (v + n) = u - v < N.$$

In particular, $(N, 0) \notin (U \times U) + \mathbb{N}$, a contradiction. ∎

Example 6.6 If an infinite semigroup S has indecomposable elements then $S \times S$ is not finitely generated as a right,left or bi S-act.

Proposition 6.7 *Let G be an infinite group. Then G does not have finitely generated diagonal right or diagonal left act. Moreover, G has finitely generated diagonal bi act if and only if G has only finitely many conjugacy classes.*

Proof. The proof of the first part is similar to that used in Example 6.5. Let U be any finite subset of G, and let U^{-1} be the corresponding set of inverses.

Choosing $h \in G$ such that $h \notin UU^{-1}$ we see that if $(h, 1_G) = (u_1, u_2)g$ for some $u_1, u_2 \in U$, $g \in G$ then

$$h = h1_G^{-1} = (u_1 g)(g^{-1}u_2^{-1}) = u_1 u_2^{-1},$$

a contradiction. Thus $(h, 1_G)$ cannot lie in $(U \times U)G$, and G does not have finitely generated diagonal right act. That G does not have finitely generated diagonal left act is proved the same way.

Suppose that G has finitely generated diagonal bi act. Then

$$G \times G = G(U \times U)G$$

for some finite subset U of G. Given $x \in G$ by hypothesis we may find $u, v \in U$ and $g, h \in G$ such that $(x, 1_G) = g(u, v)h$. Then

$$x = x1_G^{-1} = (guh)(h^{-1}v^{-1}g^{-1}) = guv^{-1}g^{-1} \in [uv^{-1}]$$

where $[uv^{-1}]$ denotes the conjugacy class containing uv^{-1} in G. Since U is finite, UU^{-1} is also finite, and so G has only finitely many conjugacy classes as required.

Conversely, if U contains an element from each conjugacy class in G then the set $U \times U$ generates $G \times G$ as a bi G-act. Indeed, given $x, y \in G$ we have $xy^{-1} \in [u]$ say, for some $u \in U$. Writing $xy^{-1} = g^{-1}ug$ we see that

$$(x, y) = g^{-1}(u, 1_G)gy \in G(U \times U)G$$

as required. ∎

We note that infinite groups with only finitely many conjugacy classes do exist - see [7] for example.

So far we have seen only negative examples.. We finish this section with an example of an infinite semigroup which does have finitely generated diagonal right act, due to Bulman-Fleming and McDowell [4].

Example 6.8 [Bulman-Fleming, McDowell] Let $T_{\mathbb{N}}$ be the monoid consisting of all mappings from $\mathbb{N}$ into $\mathbb{N}$ under composition. Then $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is both a cyclic left and right $T_{\mathbb{N}}$-act. Indeed, let α and β be functions from $\mathbb{N}$ into $\mathbb{N}$ defined by $x\alpha = 2x$, and $x\beta = 2x+1$. Then for any $(f, g) \in T_{\mathbb{N}} \times T_{\mathbb{N}}$ we have $(f, g) = (\alpha, \beta)h$ where $h : \mathbb{N} \to \mathbb{N}$ is defined by $(2m)h = mf$, $(2m+1)h = mg$. To show that $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is a cyclic left $T_{\mathbb{N}}$-act, we first choose a bijection $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and let (λ, μ) be its inverse, where λ and μ are functions from $\mathbb{N}$ to $\mathbb{N}$. Then for any $(f, g) \in T_{\mathbb{N}} \times T_{\mathbb{N}}$ we have $(f, g) = k(\lambda, \mu)$ where $k = (f, g)\psi \in T_{\mathbb{N}}$.

However, we note that $T_{\mathbb{N}}$ is not finitely generated, since it is uncountable. This example is therefore not applicable in Theorem 3.10 of Chapter 3. In the next section we consider a smaller monoid $R_{\mathbb{N}}$ of mappings, consisting of all partial recursive functions from $\mathbb{N}$ into $\mathbb{N}$. We show that $R_{\mathbb{N}}$ retains the cyclic diagonal act properties of $T_{\mathbb{N}}$, and also that it is finitely generated.

2 Partial recursive functions of one variable

Let $R_{\mathbb{N}}$ be the monoid of all partial recursive functions of one variable under composition. For various facts about the set of all partial recursive functions, see for example [5]. In the proof given in Section 1 that $T_{\mathbb{N}} \times T_{\mathbb{N}}$ is both a cyclic right and left $T_{\mathbb{N}}$-act we see that α and β are recursive, while ψ, λ and μ may be chosen to be partial recursive. Furthermore, if f and g are themselves partial recursive functions, then both h and k are, and so the proof for $T_{\mathbb{N}}$ will also work for $R_{\mathbb{N}}$. So we have the following:

Proposition 6.9 *$R_{\mathbb{N}} \times R_{\mathbb{N}}$ is both a cyclic right and a cyclic left $R_{\mathbb{N}}$-act.* ■

However, unlike $T_{\mathbb{N}}$, $R_{\mathbb{N}}$ is finitely generated.

Theorem 6.10 *The monoid $R_{\mathbb{N}}$ is finitely generated.*

PROOF. We use the fact that there exists a *universal partial recursive function* $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every partial recursive function f there is some $i \in \mathbb{N}$ such that

$$xf = (i, x)\phi \quad (x \in \mathbb{N}).$$

We let $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the partial recursive bijection defined in Example 6.8, and as before let $\lambda : \mathbb{N} \to \mathbb{N}$ and $\mu : \mathbb{N} \to \mathbb{N}$ be the partial recursive functions such that (λ, μ) is the inverse of ψ. Thus $(x\lambda, x\mu)\psi = x$ and $(x, y)\psi\lambda = x$, $(x, y)\psi\mu = y$. We define $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$x\sigma = (x\lambda, x\mu)\phi.$$

Then σ is also partial recursive. We define π and ρ as partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$ by

$$\begin{aligned} x\pi &= (0, x)\psi \\ x\rho &= (x\lambda + 1, x\mu)\psi. \end{aligned}$$

We claim that $R_{\mathbb{N}} = \langle \pi, \rho, \sigma \rangle$. Given a function $f \in R_{\mathbb{N}}$ there must exist i such that $xf = (i, x)\phi$ for each x. We first note that

$$(i, x)\psi\rho = ((i, x)\psi\lambda + 1, (i, x)\psi\mu)\psi = (i + 1, x)\psi, \tag{6.1}$$

and that

$$(i, x)\psi\sigma = \big((i, x)\psi\lambda, ((i, x)\psi\mu\big)\phi = (i, x)\phi. \tag{6.2}$$

Then we have

$$\begin{aligned} x\pi\rho^i\sigma &= (0, x)\psi\rho^i\sigma \\ &= (i, x)\psi\sigma \quad \text{(by (6.1))} \\ &= (i, x)\phi \quad \text{(by (6.2))} \\ &= xf \quad \text{(by choice of } i\text{)} \end{aligned}$$

and so $R_{\mathbb{N}}$ is finitely generated as required. ■

In fact we can do better, as the following Theorem shows.

Theorem 6.11 *The monoid $R_{\mathbb{N}}$ is 2-generated.*

PROOF. Let α, β, π, ρ and σ be defined as above. Define a function $\gamma : \mathbb{N} \to \mathbb{N}$ as follows:

$$x\gamma = \begin{cases} x+1 & \text{if } x \text{ is even} \\ y\pi & \text{if } x = 8y+1 \\ y\rho & \text{if } x = 8y+3 \\ y\sigma & \text{if } x = 8y+5 \\ 0 & \text{if } x = 8y+7. \end{cases}$$

Both α and γ are partial recursive functions. We will show that the set $X = \{\alpha, \gamma\}$ generates $R_{\mathbb{N}}$ by showing that the generators from Theorem 6.10, π, ρ and σ, lie in $\langle X \rangle$. Clearly $x\alpha\gamma = 2x+1$ and so $\beta \in \langle X \rangle$. Then $x\alpha^2\beta = 8x+1$ and so $\alpha^2\beta\gamma = \pi \in \langle X \rangle$. Similarly $x\alpha\beta^2 = 8x+3$ and $x\beta\alpha\beta = 8x+5$ and so $\rho, \sigma \in \langle X \rangle$ as required. ■

Theorem 6.12 *The monoid $R_{\mathbb{N}}$ is not finitely presented.*

PROOF. Suppose that $R_{\mathbb{N}}$ is finitely presented. Then it can be finitely presented in terms of the generators π, ρ, σ, and so we may assume that $R_{\mathbb{N}} = \langle \pi, \rho, \sigma \mid Q \rangle$ for some finite set of relations Q. Let f be a partial recursive function that is not total, and let $m \in \mathbb{N}$ be such that $xf = (m, x)\phi$ for all x (so $f = \phi_m$). For simplicity we define ϕ_n to be the function mapping x to $(n, x)\phi$. We let A be the singleton set containing f. By a corollary of the Rice-Shapiro Theorem (see [5], Theorem 2.8 and Corollary 1) the set $\{n \in \mathbb{N} : \phi_n \in A\}$ is not recursively enumerable: indeed, if it was, then any extension of f would also be in A. Now $\phi_n = \pi\rho^n\sigma$ as in the proof of Theorem 6.10. If $R_{\mathbb{N}}$ were finitely presented then there would exist a procedure that always answers yes if the two words $\pi\rho^m\sigma$ and $\pi\rho^n\sigma$ are equal in $R_{\mathbb{N}}$. This implies that the set $\{n \in \mathbb{N} : \phi_n = \phi_m\}$ is recursively enumerable. In other words, the set $\{n \in \mathbb{N} : \phi_n \in A\}$ is recursively enumerable, which gives a contradiction. ∎

3 General Properties

In this section we prove several general results about infinite semigroups with finitely generated diagonal acts.

Lemma 6.13 *If $S \times S$ is finitely generated as a right, left or bi S-act then we must have $S^2 = S$.*

PROOF. In the case where $S \times S$ is finitely generated as a right S-act, choose U finite such that $(U \times U)S^1 = S \times S$. Choose $z \in S \setminus U$. Then for any $s \in S$, by hypothesis we must have $(s, z) = (u, v)t$ for $u, v \in U$, $t \in S^1$. In fact $t \neq 1$, since $v \neq z$, and so we have $s = ut$ as required. ∎

We will use the following result frequently throughout this chapter.

Theorem 6.14 *let S be an infinite semigroup. If $S \times S$ is finitely generated as a right S-act then we may write $S \times S = (Y \times Y)S$ for some finite subset Y of S.*

PROOF. By Lemma 6.4 we know $S \times S = (U \times U)S^1$ for some finite subset U of S. Let $Y = U \cup U^2$. We will show that $S \times S = (Y \times Y)S$. Since U is finite, we may find an infinite subset $T \subseteq S \setminus U$. Let a and b be arbitrary in S. By assumption, for each $t \in T$ we may find $u_t, v_t \in U$, $s_t \in S^1$ such that

$$(a, t) = (u_t, v_t)s_t.$$

In fact, since $t \notin U$, we must have $s_t \neq 1$. The set $X = \{s_t : t \in T\}$ must be infinite, since (as subsets of S) $T \subseteq UX$. So we may choose $s_1 \in X \setminus U$, and write $a = u_1 s_1$. By Lemma 6.13 we may write $b = b_1 b_2$. By assumption we can write

$$(s_1, b_2) = (u_2, v_2)s_2,$$

for some $u_2, v_2 \in U$, $s_2 \in S^1$ where again $s_2 \neq 1$ since $s_1 \notin U$. Then we have

$$(a, b) = (u_1 u_2, b_1 v_2)s_2 \in (Y \times Y)S$$

as required. ■

We note here that the analogous result for diagonal left acts hold. Next we prove the following result, which was used in Theorem 3.10.

Proposition 6.15 *Suppose that S is a semigroup and that $S \times S$ is a finitely generated S-act. Then the direct product of n copies of S is a finitely generated S-act for each $n \in \mathbb{N}$.*

Proof. Denote by $S^{(n)}$ the direct product of n copies of S. The proof is by induction on n. Assume that $S^{(n)}$ is a finitely generated S-act, for some $n \geqslant 2$. As in the proof of Lemma 6.4, we may find a finite subset U of S such that $S^{(n)} = U^{(n)}S$. Clearly then $S^{(m)} = U^{(m)}S$ for $1 \leqslant m \leqslant n$. Let $V = U^2 \cup U$. Then V is a finite subset of S, and we claim that $S^{(n+1)} = V^{(n+1)}S$. To see this, given an element $(x_1, x_2, \dots, x_{n+1})$ of $S^{(n+1)}$ we find $(u_1, u_2, \dots u_n) \in U^{(n)}$ and $p \in S$ such that $(x_1, x_2, \dots x_n) = (u_1, u_2, \dots u_n)p$. Then we find $(u_{n+1}, u_{n+2}) \in U \times U$ and $q \in S$ such that $(p, x_{n+1}) = (u_{n+1}, u_{n+2})q$. Then

$$(u_1 u_{n+1}, u_2 u_{n+1}, \dots, u_n u_{n+1}, u_{n+2})q = (x_1, x_2, \dots, x_n, x_{n+1})$$

and $S^{(n+1)}$ is a finitely generated S-act as required. ■

Again we note that the analogous result holds for diagonal left acts.

Proposition 6.16 *Let S and T be infinite semigroups. Then $S \times T$ has finitely generated diagonal right act if and only if S and T have finitely generated diagonal right act. The analogous result holds for diagonal left acts.*

PROOF. ($\Leftarrow$) Using Theorem 6.14 we may assume that $S \times S = (U \times U)S$ and $T \times T = (V \times V)T$ for finite subsets U and V of S and T respectively. We claim that

$$(S \times T) \times (S \times T) = \big((U \times V) \times (U \times V)\big)(S \times T).$$

Indeed, given two elements (s_1, t_1) and (s_2, t_2) from $S \times T$ we may find elements $u_1, u_2 \in U$, $v_1, v_2 \in V$, $s \in S$ and $t \in T$ such that

$$\begin{aligned}(s_1, s_2) &= (u_1, u_2)s,\\ (t_1, t_2) &= (v_1, v_2)t.\end{aligned}$$

Then

$$\big((s_1, t_1), (s_2, t_2)\big) = \big((u_1, v_1), (u_2, v_2)\big)(s, t)$$

as required.

($\Rightarrow$) We may assume that

$$(S \times T) \times (S \times T) = \big((U \times V) \times (U \times V)\big)(S \times T)$$

for some finite sets U and V of S and T. Given $s_1, s_2 \in S$ we choose $t \in T$ and by hypothesis may find $u_1, u_2 \in U$, $v_1, v_2 \in V$, $s \in S$ and $t \in T$ such that

$$\big((s_1, t), (s_2, t)\big) = \big((u_1, v_1), (u_2, v_2)\big)(s, t).$$

Then $(s_1, s_2) = (u_1, u_2)s$ and so the set $U \times U$ finitely generates $S \times S$ as a right S-act. Similarly $T \times T$ also has finitely generated diagonal right act.

The corresponding result for diagonal left acts is proved in the same way. ■

Proposition 6.17 *Let S and T be infinite monoids. If S has finitely generated diagonal right act and T has finitely generated diagonal left act then $S \times T$ has finitely generated diagonal bi act.*

PROOF. Again we may assume that $S \times S = (U \times U)S$ and $T \times T = T(V \times V)$ for finite subsets U and V of S and T respectively. As before, we claim that the finite set

$$\big((U \times V) \times (U \times V)\big)$$

generates $(S \times T) \times (S \times T)$ as a bi act. Given two elements (s_1, t_1) and (s_2, t_2) from $S \times T$ we may find elements $u_1, u_2 \in U$, $v_1, v_2 \in V$, $s \in S$ and $t \in T$ such that

$$\begin{aligned} (s_1, s_2) &= (u_1, u_2)s, \\ (t_1, t_2) &= t(v_1, v_2). \end{aligned}$$

Then

$$\big((s_1, t_1), (s_2, t_2)\big) = (1_S, t)\big((u_1, v_1), (u_2, v_2)\big)(s, 1_T)$$

as required. ■

Lemma 6.18 *Let S be a semigroup with finitely generated diagonal act. Suppose that T is a subsemigroup of S such that $S \setminus T$ is an ideal of S. Then T also has finitely generated diagonal act. The analogous results hold for diagonal left and bi acts.*

PROOF. By Lemma 6.14 we may suppose that $S \times S = (U \times U)S$ for some finite subset U of S. Given $t_1, t_2 \in T \leqslant S$ we must have

$$(t_1, t_2) = (u_1, u_2)s$$

for some $u_1, u_2 \in U$, $s \in S$. Since $S\backslash T$ is an ideal, we must also have $u_1, u_2, s \in T$. Therefore $T \times T = (V \times V)T$ where $V = T \cap U$. The results for diagonal left and bi acts are proved in the same way. ■

4 Further examples of finitely generated diagonal acts

Theorem 6.19 *There exists an infinite finitely presented monoid P such that $P \times P$ is a cyclic right P-act and a cyclic left P-act.*

PROOF. We construct such a finitely presented monoid P which has $R_{\mathbb{N}}$ as a homomorphic image. Let $A = \{p, r, s, m\}$ be an alphabet, the letters p, r, s, m representing the generators π, ρ, σ, μ of $R_{\mathbb{N}}$ respectively, and let $f : A^* \to R_{\mathbb{N}}$ be the corresponding epimorphism. (As usual, A^* denotes the free monoid on A consisting of all words over A including the empty word 1.) Let $a, b \in A^*$ be such that $af = \alpha$, $bf = \beta$. For each $x \in A$ let $u_x, v_x \in A^*$ be such that

$$\begin{aligned}(\alpha,\beta)[u_x f] &= ((xf)\alpha, \beta)\\ (\alpha,\beta)[v_x f] &= (\alpha, (xf)\beta).\end{aligned}$$

Also, let $w \in A^*$ be such that $(\alpha, \beta)[wf] = (1_{\mathbb{N}}, 1_{\mathbb{N}})$, where $1_{\mathbb{N}}$ denotes the identity mapping on $\mathbb{N}$. We now define P_1 to be the monoid defined by the presentation

$$\begin{aligned}\langle\, p, r, s, m \mid au_x = xa, bu_x = b, av_x = a, bv_x = xb \quad (x \in \{p, r, s, m\})\\ aw = bw = 1\,\rangle.\end{aligned} \tag{6.3}$$

Clearly, P_1 is finitely presented and has $R_{\mathbb{N}}$ as a homomorphic image, so that it is infinite. We now prove that $P_1 \times P_1$ is a cyclic right P_1-act. Indeed, for any

$w_1, w_2 \in A^*$, with $w_1 = x_1x_2\dots x_k$, $w_2 = y_1y_2\dots y_n$ $(x_i, y_j \in A)$ we have

$$\begin{aligned} & au_{x_1}u_{x_2}\dots u_{x_k}v_{y_1}\dots v_{y_n}w = x_1au_{x_2}\dots u_{x_k}v_{y_1}\dots v_{y_n}w = \dots \\ = \quad & x_1\dots x_k av_{y_1}\dots v_{y_n}w = x_1\dots x_k av_{y_2}\dots v_{y_n}w = \dots = x_1\dots x_k aw \\ = \quad & x_1\dots x_k = w_1 \end{aligned}$$

as a consequence of defining relations, and similarly

$$bu_{x_1}\dots u_{x_k}v_{y_1}\dots v_{y_n}w = w_2.$$

Therefore $P_1 \times P_1$ is generated (as a right P_1-act) by (a, b).

One can now use the same technique and the fact that $R_{\mathbb{N}} \times R_{\mathbb{N}}$ is a cyclic left $R_{\mathbb{N}}$-act to add a further 18 relations to (6.3), obtaining a monoid P such that $R_{\mathbb{N}}$ is a homomorphic image of P and $P \times P$ is both a cyclic right P-act and a cyclic left P-act. ■

Our next construction is aimed at demonstrating the independence of properties of $S \times S$ as a right S-act from those of $S \times S$ as a left S-act.

Given a semigroup S we construct a new semigroup $C(S)$ as follows. Let $S^{(1)}$ and $S^{(2)}$ be disjoint sets in 1-1 correspondence with S, where $s \leftrightarrow s^{(1)} \leftrightarrow s^{(2)}$ are bijections, and let $C(S) = S^{(1)} \cup S^{(2)}$. We define multiplication on $C(S)$ as follows:

$$\begin{aligned} s^{(1)}t^{(1)} &= (st)^{(1)}, & s^{(1)}t^{(2)} &= t^{(2)}, \\ s^{(2)}t^{(1)} &= (st)^{(2)}, & s^{(2)}t^{(2)} &= t^{(2)}. \end{aligned}$$

This turns $C(S)$ into a semigroup; in [8], $C(S)$ is called the constant extension of S. It is easy to see that $C(S)$ is a monoid if and only if S is a monoid. Now we prove the following facts about $C(S)$.

Theorem 6.20 *Let S be any semigroup, and let $C = C(S)$. Then*

(i) *C is finitely generated if and only if S is finitely generated;*

(ii) *$C \times C$ is a finitely generated right C-act if and only if $S \times S$ is a finitely generated right S-act;*

(iii) *if S is infinite, then $C \times C$ is not a finitely generated left C-act.*

PROOF. (i) Suppose that $S = \langle X \rangle$, and let $X^{(1)}$ and $X^{(2)}$ be the copies of X in $S^{(1)}$ and $S^{(2)}$ respectively. We show that $C = \langle X^{(1)} \cup X^{(2)} \rangle$. Indeed, if $t \in C$ with $t = s^{(i)}$, and if $s = x_1 \dots x_n$ $(x_j \in X)$ then

$$t = x_1^{(i)} x_2^{(1)} \dots x_n^{(1)}.$$

The converse follows from the fact that $S^{(1)} \cong S$ and that $C \setminus S^{(1)} = S^{(2)}$ is an ideal of C.

(ii) It is easy to check that if $S \times S = (U \times U)S^1$ then $C \times C = (V \times V)C^1$ where $V = U^{(1)} \cup U^{(2)}$. Thus if $S \times S$ is finitely generated as a right S-act then $C \times C$ is finitely generated as a right C-act. For the converse, we note that if the C-act $C \times C$ is generated by a set $V \times V$, then the S-act $S \times S$ is generated by $U \times U$ where $U = \{u \in S : u^{(1)} \in V\}$.

(iii) Suppose $C \times C = C^1\big((U^{(1)} \cup U^{(2)}) \times (U^{(1)} \cup U^{(2)})\big)$ for some finite $U \subseteq S$. Choose $p, q \in S \setminus U$. By hypothesis we can write $(p^{(1)}, q^{(2)}) = t^{(i)}(u^{(j)}, v^{(k)})$ for some $t \in S$, $u, v \in U$, $i, j, k \in \{1, 2\}$. From the way that multiplication between elements of $S^{(1)}$ and $S^{(2)}$ in C is defined, we see that $p^{(1)} = t^{(i)} u^{(j)}$ implies $i = j = 1$, and then $q^{(2)} = t^{(1)} v^{(k)}$ implies that $k = 2$. But $t^{(1)} v^{(2)} = v^{(2)}$, and so $q = v$. Thus every element of S must lie in U, and U cannot be finite. Thus $C \times C$ is not a finitely generated left C-act. ∎

Corollary 6.21 *The monoid $C = C(R_{\mathbb{N}})$ is finitely generated. Furthermore, $C \times C$ is finitely generated as a right C-act, but is not finitely generated as a left C-act.* ∎

We now describe another semigroup construction. Given a semigroup S we construct $D(S)$ to be the direct product of S with its opposite, S'. The elements of S' are in 1-1 correspondence $s \leftrightarrow s'$ with S, and multiplication is given by $s't' = (ts)'$. Obviously S' (and hence $D(S)$ as well) is a monoid if and only if S is a monoid. Also S' is finitely generated (respectively finitely presented) if and only if S is finitely generated (finitely presented). We now prove the following facts about $D(S)$.

Theorem 6.22 *Let S be any semigroup, and let $D = D(S)$.*

(i) *If $D \times D$ is a finitely generated right (or left) D-act then $S \times S$ is both a finitely generated right S-act and a finitely generated left S-act.*

(ii) *If S is a monoid and $S \times S$ is a finitely generated right S-act then $D \times D$ is a finitely generated bi D-act.*

Proof. (i) Suppose $D \times D = (U \times U)D^1$, where U is finite. We may assume that $U = V \times V'$ where V is some finite subset of S, and V' is the corresponding finite subset in S'. We claim that $S \times S$ is finitely generated by the set $V \times V$ both as a right and a left S-act. Given $p, q \in S$, not both in V, the hypothesis allows us to write

$$\big((p,p'),(q,q')\big) = \big((v_1,v_2'),(v_3,v_4')\big)(r,s')$$

for $r, s \in S$, $v_i \in V$. Thus

$$(p,p') = (v_1,v_2')(r,s') \tag{6.4}$$

$$(q,q') = (v_3,v_4')(r,s'). \tag{6.5}$$

Equating first components in (6.4) and (6.5) we see that $p = v_1 r$, $q = v_3 r$, and so $(p,q) = (v_1,v_3)r \in (V \times V)S$. Equating second components in (6.4) we see that $p' = v_2' s'$ in S', and so $p = s v_2$. Similarly from (6.5) we obtain $q = s v_4$, and so $(p,q) = s(v_2,v_4) \in S(V \times V)$.

(ii) Suppose that $S \times S = (V \times V)S$, for some finite subset V of S. Then we claim that $D \times D = D(U \times U)D$, where $U = V \times V'$ is finite, and so $D \times D$ is a finitely generated bi D-act. To see this, we take two arbitrary elements (a, b') and (c, d') of D, where $a, b, c, d \in S$. Since $S \times S$ is finitely generated as a right S-act, we may find $s, t \in S$ and $v_1, v_2, v_3, v_4 \in V$ such that $(a, c) = (v_1, v_3)s$ and $(b, d) = (v_2, v_4)t$. Then $b' = t'v_2'$ and $d' = t'v_4'$ and so we have

$$(1, t')\big((v_1, v_2'), (v_3, v_4')\big)(s, 1') = \big((a, b'), (c, d')\big)$$

as required. ■

Corollary 6.23 *The monoid $D = D(C(R_{\mathbb{N}}))$ is finitely generated. Furthermore, $D \times D$ is a finitely generated bi D-act, but is not finitely generated as either a left or a right D-act.* ■

5 Power semigroups

In this section we investigate links between diagonal acts and power semigroups. We start by defining the power semigroup.

Definition 6.24 Let S be a semigroup. The *power semigroup* of S, denoted $\mathcal{P}_f(S)$, is the set of all finite subsets of S with semigroup operation

$$AB = \{ab : a \in A, b \in B\}.$$

It is clear that if S is commutative, so is $\mathcal{P}_f(S)$. In the following example this is the case.

Example 6.25 The power semigroup $\mathcal{P}_f(\mathbb{Z})$ is not finitely generated, where $\mathbb{Z}$ is the infinite cyclic group (using addition as the semigroup operation). To show this, we take a finite number of finite subsets $A_1, \ldots A_r$ of $\mathbb{Z}$, and show that these do not suffice even to generate all two element subsets of $\mathbb{Z}$. First we note that if $i < j$ and $k < l$ then

$$\{a^i, a^j\} + \{a^k, a^l\} = \{a^{i+k}, a^{i+l}, a^{j+k}, a^{j+l}\},$$

where $i+k < i+l < j+l$. Thus if $|AB| = 2$ then either $|A| = 1$ or $|B| = 1$. Since the power semigroup is commutative, we may assume that $|B| = 1$. Clearly then we must have $|A| = 2$. Now if $A = \{a^i, a^j\}$ and $B = \{a^k\}$ then $AB = \{a^{i+k}, a^{j+k}\}$. The difference between the two elements in A is preserved when multiplying by the one element set B. Because of this, if we have only a finite number of two element sets to choose from, we can only generate two element sets where the difference between the two elements takes a finite number of values. Thus $\mathcal{P}_f(\mathbb{Z})$ is not finitely generated. ∎

Theorem 6.26 *If $\mathcal{P}_f(S)$ is finitely generated, then S must be finitely generated.*

PROOF. Suppose that $\mathcal{P}_f(S)$ is finitely generated with generators the finite sets $A_1, \ldots, A_n$. Then S is finitely generated by the set $\bigcup_{i=1}^n A_i$. Given $s \in S$, by hypothesis we have

$$\{s\} = A_{j_1} A_{j_2} \ldots A_{j_r}$$

and so $s = a_1 \ldots a_r$ for any $a_i \in A_{j_i}$. ∎

While it is not true that if S is finitely generated, then the power semigroup $\mathcal{P}_f(S)$ is also finitely generated, the converse does hold.

Theorem 6.27 *Let S be any semigroup. If $\mathcal{P}_f(S)$ is finitely generated then $S \times S$ is a finitely generated bi S-act.*

PROOF. Suppose that $\mathcal{P}_f(S)$ is finitely generated by the finite sets $A_1, \ldots, A_n$. We will show that the bi S-act $S \times S$ is generated by the (finite) set $U \times U$ where

$U = \bigcup_{i=1}^{n} A_i$. Let $p, q \in S$ be arbitrary, and write

$$\{p, q\} = A_{j_1} A_{j_2} \dots A_{j_r}.$$

In particular, we have $p = x_1 x_2 \dots x_r$, $q = y_1 y_2 \dots y_r$ for some $x_i, y_i \in A_{j_i}$. Thus we have

$$\{p, q\} = B_1 B_2 \dots B_r \tag{6.6}$$

where $B_i = \{x_i, y_i\} \subseteq A_{j_i} \subseteq U$ has at most two elements. Clearly, there must exist at least one set, B_m say, with precisely two elements. Consider the sets

$$\begin{aligned} X &= B_1 \dots B_{m-1} \{x_m\} B_{m+1} \dots B_r \\ Y &= B_1 \dots B_{m-1} \{y_m\} B_{m+1} \dots B_r. \end{aligned}$$

If $|X| = 2$ then B_m can be replaced by just $\{x_m\}$ with (6.6) remaining valid. Similarly if $|Y| = 2$ then B_m can be replaced by $\{y_m\}$. If $|X| = |Y| = 1$, then all B_i with $i \neq m$ can be replaced by one element sets $\{x_i\}$. Repeating this, if necessary, we obtain

$$\{p, q\} = \{z_1\} \dots \{z_{k-1}\} \{x_k, y_k\} \{z_{k+1}\} \dots \{z_r\}$$

for some k $(1 \leq k \leq r)$ and some $z_i \in B_i \subseteq U$. Thus we have either

$$(p, q) = z_1 \dots z_{k-1} (x_k, y_k) z_{k+1} \dots z_r \in S^1 (U \times U) S^1$$

or

$$(p, q) = z_1 \dots z_{k-1} (y_k, x_k) z_{k+1} \dots z_r \in S^1 (U \times U) S^1,$$

completing the proof. ∎

Sadly, the converse to Theorem 6.27 fails - the monoid $D = D(C(R_{\mathbb{N}}))$ constructed in Section 4 has finitely generated diagonal bi act but $\mathcal{P}_f(D)$ is not finitely generated, as we will show in Proposition 6.31. First we make a couple of observations about power monoids of this type. Let S be an infinite monoid. We note that by construction, elements of $D(C(S))$ are of the form $(s^{(i)}, t^{(j)})$, for $s, t \in S$, $i, j \in \{1, 2\}$. We will refer to such an element as being of *type* $[i, j]$. So there are four 'types' of element in D. It is easy to see that multiplying an element of type $[i, j]$ by one of type $[k, l]$ yields an element of type $[m, n]$ where $m = \max\{i, k\}$ and $n = \max\{j, l\}$.

Lemma 6.28 *Let S be an infinite monoid, let $D = D(C(S))$ and let $\mathcal{P}_f(D)$ be the power monoid of D. Then the subset M_1 of $\mathcal{P}_f(D)$, consisting of all finite subsets of D containing at least one element of type $[1, 1]$, is a submonoid of $\mathcal{P}_f(D)$. Moreover, the complement of M_1 in $\mathcal{P}_f(S)$ is an ideal.*

PROOF. Both assertions follow from the fact that the product of elements st in $D(S)$ is of type $[1, 1]$ if and only if both s and t are of type $[1, 1]$. ∎

Lemma 6.29 *Let M_1 be the monoid defined in Lemma 6.28, and let M be the subset of M_1 consisting of all finite subsets of D containing at least one element of type $[1, 1]$, and no type $[1, 2]$ or $[2, 1]$ elements. Then M is a submonoid of M_1, whose complement in M_1 is again an ideal.*

PROOF. It is clear that M is a submonoid. Suppose that X is an element of $M_1 \setminus M$, and Y an element of M. Then Y, as an element of M_1, must contain at least one element of type $[1, 1]$, y say. Also, X must contain at least one element either of type $[1, 2]$ or $[2, 1]$, call this x. Then both xy and yx have the same type as x, and so neither XY or YX lie in M. Thus the complement of M in M_1 is indeed an ideal as required. ∎

Corollary 6.30 *Let S be an infinite monoid, let $D = D(C(S))$ and let $\mathcal{P}_f(D)$ be the power monoid of D. Let M be the monoid defined in Lemma 6.29 above.*

If $\mathcal{P}_f(D)$ is finitely generated then M is also finitely generated.

PROOF. This is simply a double application of Theorem 1.6, using Lemmas 6.28 and 6.29 above. ■

Proposition 6.31 *Let S be an infinite monoid and let $D = D(C(S))$. The power monoid $\mathcal{P}_f(D)$ is not finitely generated.*

PROOF. By Corollary 6.30 it will suffice to show that the monoid M consisting of all finite subsets of D with at least one type $[1,1]$ element, and no type $[1,2]$ or $[2,1]$ elements cannot be finitely generated. Suppose for a contradiction that Z_i $(i \in I_0)$ is a finite generating set for M. Let N be an integer greater than the number of type $[2,2]$ elements in any Z_i $(i \in I_0)$. For convenience, we expand the index set I_0 to I to include indices for all possible subsets of D consisting of finitely many type $[1,1]$ elements, at most $N-1$ type $[2,2]$ elements, and no other element types. Our index set I is now infinite, and obviously if $\{Z_i : i \in I_0\}$ generates M then certainly $\{Z_i : i \in I\}$ does.

Let $s_0, \dots, s_N$ be distinct elements of S and define

$$X = \{(s_0^{(1)}, s_0^{(1)})\} \cup \{(s_i^{(2)}, s_i^{(2)}) : 1 \leqslant i \leqslant N\},$$

a set with one type $[1,1]$ element and N type $[2,2]$ elements. Let

$$Z_1 Z_2 \dots Z_r$$

be a minimal length expression for X written as a product of generators from the infinite set $\{Z_i : i \in I\}$. Since X is not a generator, we must have $r \geqslant 2$.

Let $Z = Z_1 Z_2$. We will show that we may replace Z by another set Z' from $\{Z_i : i \in I\}$ while leaving the product $Z Z_3 \dots Z_r$ unchanged. This contradiction will give us the result.

We may write

$$\begin{aligned} Z_i &= \{(a_j, b_j), (c_k, d_k) : j \in J, k \in K\} \\ Z_{i+1} &= \{(e_p, f_p), (g_q, h_q) : p \in P, q \in Q\} \end{aligned}$$

where the elements (a_j, b_j) and (e_p, f_p) are of type $[1, 1]$, the elements (c_k, d_k) and (g_q, h_q) are of type $[2, 2]$, and J, K, P and Q are simply finite indexing sets. We do not discount the possibility that K and Q may be empty. Then $Z_1Z_2 = W \cup U$ where W is some finite set of type $[1, 1]$ elements (which we will not concern ourselves with) and

$$U = \{(g_q, h_qb_j), (c_ke_p, d_k), (g_q, d_k) : j \in J, k \in K, p \in P, q \in Q\},$$

a set consisting of the type $[2, 2]$ elements. (If some index set is empty, we simply ignore any element with an index from that set). The elements of U fall into three (not necessarily non-empty) sets:

$$\begin{aligned} U_1 &= \{(g_q, h_qb_j) : j \in J, q \in Q\}; \\ U_2 &= \{(c_ke_p, d_k) : k \in K, p \in P\}; \\ U_3 &= \{(g_q, d_k) : k \in K, q \in Q\}. \end{aligned}$$

Suppose we have elements (x_1, y) and (x_2, y) of type $[2, 2]$ in Z, having identical second components. Then for any $z_n \in Z_n$ $(3 \leqslant n \leqslant r)$ the (type $[2, 2]$) elements $(x_1, y)z_3 \dots z_r$ and $(x_2, y)z_3 \dots z_r$ also have the same second component. Since distinct elements in X have distinct second components, these elements must in fact be the same. So one of the elements, (x_2, y) say, is redundant in generating X, and we may remove it from Z while leaving the product $ZZ_3 \dots Z_r$ unchanged. The same obviously holds for elements with identical first components.

Now, if U_3 is non-empty, then we may eliminate the sets U_1 and U_2, since their elements share common components with elements in U_3. If U_3 is empty, then one (or both) of the index sets Q or K must be empty, and so at least one of U_1 and U_2 is also empty. Therefore, we must have that

$$(U_i \cup W)Z_3 \dots Z_r = ZZ_3 \dots Z_r = X$$

for some $1 \leqslant i \leqslant 3$. We assume that $i = 1$, the other cases are treated similarly. If U_1 is empty, since W itself is already a member of the expanded generating set $\{Z_i : i \in I\}$ we can write $X = WZ_3 \dots Z_r$, a word of length $r-1$ for X. If U_1 is non-empty, then both Q and J must be non-empty. Fix an index $1 \in J$, and consider $V = \{(g_q, h_q b_1) : q \in Q\}$. Now, every element of U_1 shares a common first component with some element in V, and so $(V \cup W)Z_3 \dots Z_r = X$. Also, $|V| = |Q|$ where $|Q|$ is the number of type $[2,2]$ elements in Z_2, and so $|V| < N$. Thus $V \cup W$ is an element of $\{Z_i : i \in I\}$ and we have found a word for X of length $r-1$ - a contradiction. ∎

Corollary 6.32 *The power semigroup $\mathcal{P}_f(D(C(R_{\mathbb{N}}))$ is not finitely generated, although $D(C(R_{\mathbb{N}}))$ does have finitely generated diagonal bi-act.* ∎

As the last example showed, having finitely generated diagonal bi act is not sufficient for $\mathcal{P}_f(S)$ to be finitely generated. We now give a stronger condition that is sufficient.

Theorem 6.33 *Let S be any finitely generated semigroup such that $S \times S$ is a cyclic right (or left) S-act. Then $\mathcal{P}_f(S)$ is finitely generated.*

Proof. Suppose $S = \langle X \rangle$, and that $S \times S = (a,b)S$. We will prove that $\mathcal{P}_f(S)$ is generated by the set $Y = \big\{\{a,b\}\big\} \cup \big\{\{x\} : x \in X\big\}$. Suppose $P \in \mathcal{P}_f(S)$. By induction on $|P|$ we prove that P can be written as a product of sets from Y. If $|P| = 1$ then P is easily seen to be a product of singleton sets. Suppose $P =$

$\{p_1, \ldots, p_{n+1}\}$, and that all sets with at most n elements can be generated. Since $S \times S = (a,b)S$ we may choose elements $q_1, \ldots, q_n$ such that $(a,b)q_i = (p_i, p_i)$ for $1 \leqslant i \leqslant n-1$ and $(a,b)q_n = (p_n, p_{n+1})$. Then we have

$$\{p_1, \ldots, p_{n+1}\} = \{a, b\}\{q_1, \ldots, q_n\}$$

and our proof by induction is completed. ∎

Corollary 6.34 *$\mathcal{P}_f(R_\mathbb{N})$ is finitely generated.*

We might hope that the converse to Theorem 6.33 held, i.e. that if $\mathcal{P}_f(S)$ is finitely generated then $S \times S$ is a cyclic right or left S-act. In fact this is not the case, as our next example shows.

Proposition 6.35 *Let $C = C(R_\mathbb{N})$ as in Corollary 6.21. Then $\mathcal{P}_f(C)$ is finitely generated, but $C \times C$ is not a cyclic right C-act, and is not even finitely generated as a left C-act.*

PROOF. That $C \times C$ is not finitely generated as a left C-act, but is finitely generated as a right C-act follows from Theorem 6.20. Suppose $C \times C$ were cyclic as a right C-act, i.e. that $C \times C = (a^{(i)}, b^{(j)})C$ for $i, j \in \{1, 2\}$, $a, b \in R_\mathbb{N}$. Then for generating $(p^{(1)}, q^{(1)})$ to be possible we would need $i = j = 1$, which would make generating $(p^{(1)}, q^{(2)})$ impossible. Thus $C \times C$ is not a cyclic right C-act.

We saw in Proposition 6.9 that $R_\mathbb{N} \times R_\mathbb{N}$ is a cyclic right $R_\mathbb{N}$-act, with generator (α, β), and also a cyclic left $R_\mathbb{N}$-act, with generator (λ, μ). By Theorem 6.10 $R_\mathbb{N}$ is finitely generated, by the set X say. We will show that the finite set

$$\big\{\{\xi^{(i)}\} : \xi \in X, i \in \{1,2\}\big\} \cup \big\{\{\lambda^{(1)}, \mu^{(1)}\}, \{\alpha^{(2)}, \beta^{(1)}\}\big\}$$

generates $\mathcal{P}_f(C)$. Clearly we can generate all singleton sets - if $f = \xi_1 \ldots \xi_r \in R_\mathbb{N}$ then $\{f^{(1)}\} = \{\xi_1^{(1)}\} \ldots \{\xi_r^{(1)}\}$ and $\{f^{(2)}\} = \{\xi_1^{(2)}\}\{\xi_2^{(1)}\} \ldots \{\xi_r^{(1)}\}$. Since $R_\mathbb{N} \times R_\mathbb{N}$

is a cyclic left $R_{\mathbb{N}}$-act we may prove, as in Theorem 6.33, that any subset P of $R_{\mathbb{N}}$ of n elements may be written as $P = \{f\}\{\lambda, \mu\}^{n-1}$ where $f \in R_{\mathbb{N}}$. Also, if $P = \{p_1, \dots, p_n\}$ then we may let $Q = \{q_1, \dots, q_n\}$ where q_i is chosen so that $q_i(\lambda, \mu) = (p_i, p_i)$, and we see that $P = Q\{\lambda, \mu\}$. So we may actually write an n element subset as $\{f'\}\{\lambda, \mu\}^m$ for any $m \geqslant n-1$.

Now given a finite subset Z of C we may write $Z = U \cup V$ where $U \subseteq R_{\mathbb{N}}^{(1)}$ and $V \subseteq R_{\mathbb{N}}^{(2)}$ are both finite. By the above, we may write

$$\begin{aligned} U &= \{f^{(1)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m \\ V &= \{g^{(2)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m \end{aligned}$$

for some $f, g \in R_{\mathbb{N}}$, $m \in \mathbb{N}$. Then it is easy to see that

$$Z = \{f^{(1)}\}\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\}\{\lambda^{(1)}, \mu^{(1)}\}^m.$$

Thus all that remains to check is that we may generate the sets $\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\}$ for $g \in R_{\mathbb{N}}$. Choosing $h \in R_{\mathbb{N}}$ such that $(\alpha, \beta)h = (g, 1_{\mathbb{N}})$ we see that $\{g^{(2)}, 1_{\mathbb{N}}^{(1)}\} = \{\alpha^{(2)}, \beta^{(1)}\}\{h^{(1)}\}$. Thus $\mathcal{P}_f(C)$ is finitely generated as required. ∎

6 Final Remarks

Although we have given several results regarding the link between diagonal acts and power semigroups, we are still far from knowing the whole story. Various questions still remain unanswered. For example, does the generalization of Theorem 6.33 hold, i.e. is it true that when $S \times S$ is a finitely generated left or right S-act, or even a cyclic bi S-act, then $\mathcal{P}_f(S)$ is finitely generated? Theorem 6.27 tells us that if $P_f(S)$ is finitely generated then $S \times S$ must be finitely generated as a bi S-act, but the converse fails in general - as shown in Proposition 6.31, $\mathcal{P}_f(S)$ need not be finitely generated when $S \times S$ is finitely generated as a bi

S-act. In fact we do not have a single example of a semigroup S whose power semigroup is finitely generated but which has neither finitely generated left or right diagonal act. One thing we have not investigated here at all is the question of finite presentability of power semigroups. If S is infinite, can $\mathcal{P}_f(S)$ ever be finitely presented?

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